The University of Georgia
Department of Physics and Astronomy

Prelim Exam
August 12, 2022

Part II (problems 5 and 6)
2:00 pm – 4:00 pm

Instructions:

- Start each problem on a new sheet of paper. Write the problem number on the top left of each page and your pre-arranged prelim ID number (but ***not*** your name) on the top right of each page.
- Leave margins for stapling and photocopying.
- Write only on *one side* of the paper. Please ***do not*** write on the back side.
- If not advised otherwise, derive the mathematical solution for a problem from basic principles or general laws (Newton's laws, the Maxwell equations, the Schrödinger equation, etc.).
- You may use a calculator for basic operations only (i.e., not for referring to notes stored in memory, symbolic algebra, symbolic and numerical integration, etc.) The use of cell phones, tablets, and laptops is not permitted.
- Show your work and/or explain your reasoning in *all* problems, as the graders are not able to read minds. Even if your final answer is correct, not showing your work and reasoning will result in a *substantial* penalty.
- Write your work and reasoning in a neat, clear, and logical manner so that the grader can follow it. Lack of clarity is likely to result in a substantial penalty.

Problem 5: Quantum Mechanics (KK)

Consider the one-dimensional particle in a box problem. The box potential is zero inside the box and infinite outside and at the walls, located at $x = 0$ and $x = w$: The mass of the point particle is m.

1. Write down the time independent Schrödinger equation in one dimension for this one-dimensional particle in a box problem.
2. What are the solutions of the Schrödinger equation for the wavefunctions and energy levels?
3. Show explicitly that your answers for part 2 satisfy the Schrödinger equation.
4. Why is the ground state is $n = 1$ and not $n = 0$?
5. What is n for first excited state? and where is the positions of maximum probability density for particle in the first excited state?

Solution:

1. The time independent Schrödinger equation in one dimension:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x) = E\Psi(x).$$

2. $\Psi_n(x) = A_n sin(k_n x)$, $\quad E_n = \frac{\hbar^2 k_n^2}{2m}$, $\quad k_n = \frac{n\pi}{w}$.

3. Substitute wavefunction into the equation and differentiate:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}A_n sin(k_n x) = \frac{\hbar^2 k_n^2}{2m}A_n sin(k_n x) = E_n\Psi(x).$$

4. If $n = 0$, $\Psi_{n=0} = 0$ and this wavefunction is not normalizable.

5. First excited state has $n = 2$ and a wavefunction of $\Psi_2(x) = A_2 sin(\frac{2\pi}{w}x)$.

The probability density is $[A_2 sin(\frac{2\pi}{w}x)]^2 = [A_2]^2[sin(\frac{2\pi}{w}x)]^2$.

There are two peaks at $x = \frac{w}{4}$ and $x = \frac{3w}{4}$.

<u>Problem 6: Quantum Mechanics</u> (HHM)

Consider a quantum system with a set of energy eigenstates $|E_j\rangle$. The system is in the state

$$|\psi\rangle = \frac{1}{\sqrt{30}}|E_1\rangle + \frac{2}{\sqrt{30}}|E_2\rangle + \frac{3}{\sqrt{30}}|E_3\rangle + \frac{4}{\sqrt{30}}|E_4\rangle$$

where the energies are given by $E_n = nE_1$.

(a) Find the probabilities for measuring the energy eigenvalues.

(b) Find the expectation value of the energy.

(c) Find the uncertainty of the energy.

Solution:

5.21 The probabilities are

$$\mathcal{P}_{E_1} = \left|\langle E_1|\psi\rangle\right|^2 = \left|\langle E_1|\left(\tfrac{1}{\sqrt{30}}|E_1\rangle + \tfrac{2}{\sqrt{30}}|E_2\rangle + \tfrac{3}{\sqrt{30}}|E_3\rangle + \tfrac{4}{\sqrt{30}}|E_4\rangle\right)\right|^2 = \tfrac{1}{30}$$

$$\mathcal{P}_{E_2} = \left|\langle E_2|\psi\rangle\right|^2 = \left|\langle E_2|\left(\tfrac{1}{\sqrt{30}}|E_1\rangle + \tfrac{2}{\sqrt{30}}|E_2\rangle + \tfrac{3}{\sqrt{30}}|E_3\rangle + \tfrac{4}{\sqrt{30}}|E_4\rangle\right)\right|^2 = \tfrac{4}{30}$$

$$\mathcal{P}_{E_3} = \left|\langle E_3|\psi\rangle\right|^2 = \left|\langle E_3|\left(\tfrac{1}{\sqrt{30}}|E_1\rangle + \tfrac{2}{\sqrt{30}}|E_2\rangle + \tfrac{3}{\sqrt{30}}|E_3\rangle + \tfrac{4}{\sqrt{30}}|E_4\rangle\right)\right|^2 = \tfrac{9}{30}$$

$$\mathcal{P}_{E_4} = \left|\langle E_4|\psi\rangle\right|^2 = \left|\langle E_4|\left(\tfrac{1}{\sqrt{30}}|E_1\rangle + \tfrac{2}{\sqrt{30}}|E_2\rangle + \tfrac{3}{\sqrt{30}}|E_3\rangle + \tfrac{4}{\sqrt{30}}|E_4\rangle\right)\right|^2 = \tfrac{16}{30}$$

The average value of the energy is

$$\langle E\rangle = \sum_{n=1}^{\infty}\mathcal{P}_{E_n}E_n = \sum_{n=1}^{\infty}\left|\langle E_n|\psi\rangle\right|^2 nE_1 = E_1\left(\tfrac{1}{30}1 + \tfrac{4}{30}2 + \tfrac{9}{30}3 + \tfrac{16}{30}4\right) = \frac{10}{3}E_1$$

To find the uncertainty, we need the average of the squares:

$$\langle E^2\rangle = \sum_{n=1}^{\infty}\mathcal{P}_{E_n}E_n^2 = \sum_{n=1}^{\infty}\left|\langle E_n|\psi\rangle\right|^2 n^2E_1^2 = E_1^2\left(\tfrac{1}{30}1 + \tfrac{4}{30}4 + \tfrac{9}{30}9 + \tfrac{16}{30}16\right) = \frac{59}{5}E_1$$

The uncertainty of the energy is

$$\Delta E = \sqrt{\langle E^2\rangle - \langle E\rangle^2} = E_1\sqrt{\left(\tfrac{59}{5}\right) - \left(\tfrac{10}{3}\right)^2} = E_1\sqrt{\tfrac{31}{45}} \cong 0.83\ E_1$$