The University of Georgia
Department of Physics and Astronomy

Prelim Exam
August 12, 2022

Part I (problems 1, 2, 3, and 4)
9:00 am – 1:00 pm

Instructions:

- Start each problem on a new sheet of paper. Write the problem number on the top left of each page and your pre-arranged prelim ID number (but ***not*** your name) on the top right of each page.
- Leave margins for stapling and photocopying.
- Write only on *one side* of the paper. Please ***do not*** write on the back side.
- If not advised otherwise, derive the mathematical solution for a problem from basic principles or general laws (Newton's laws, the Maxwell equations, the Schrödinger equation, etc.).
- You may use a calculator for basic operations only (i.e., not for referring to notes stored in memory, symbolic algebra, symbolic and numerical integration, etc.) The use of cell phones, tablets, and laptops is not permitted.
- Show your work and/or explain your reasoning in *all* problems, as the graders are not able to read minds. Even if your final answer is correct, not showing your work and reasoning will result in a *substantial* penalty.
- Write your work and reasoning in a neat, clear, and logical manner so that the grader can follow it. Lack of clarity is likely to result in a substantial penalty.

Problem 1: Classical Mechanics

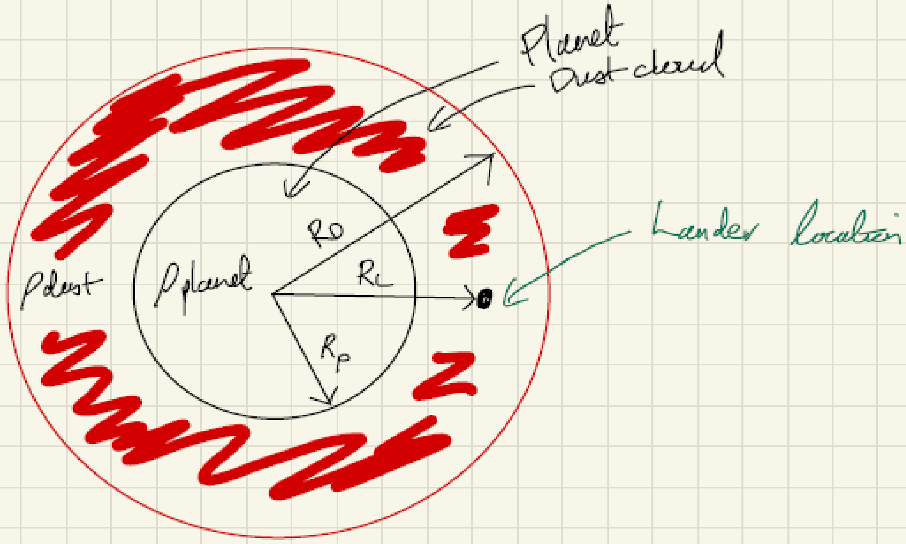
You are an interplanetary explorer, and you have just discovered a new exoplanet, WG-4559-b. It has a mass of 7×10^{24} kg, and a radius of 6,400 km. It has very high density dust storms that cover the whole planet, completely obscuring the surface.

Measurements report that the dust cloud has a uniform density of 0.1 g cm^{-3}, and it extends 2,400 km above the surface.

Your first officer makes a calculation error while attempting to teleport (transport across a distance instantly) a 65 kg robot lander to the surface of WG-4559, and the robot materialises 1300 km above the ground, inside the planet-wide dust storm.

(a) What is the gravitational force acting on the robot?

(b) It is rate to withstand 1 million Newtons of force. Does it survive?



Students may solve using mass of planet or density of planet, both are fine. Solving using planet mass is as follows

The total force acting on the lander is:

$$F = \frac{G\left[M_{planet} + M_{dust}(R<R_L)\right]\cdot M_{Lander}}{R_L^2}$$

Where $M_{dust}(R<R_L)$ is the mass of the dust at $R<R_L$.

$$M_{dust} = \rho_{dust} \times Volume_{dust} \qquad \rho_{dust} = 0.1\ g\ cm^{-3} = 100\ kg\ m^{-3}$$

$$= \rho\left[\frac{4}{3}\pi R_L^3 - \frac{4}{3}\pi R_p^3\right]$$

$$= 100\left[\frac{4}{3}\pi\left(7.7\times10^6\right)^3 - \frac{4}{3}\pi\left(6.4\times10^6\right)^3\right]$$

$$M_{dust} = 8.14\times10^{22}\ kg$$

$$F = \frac{G\left[M_{planet} + M_{dust}(R < R_L)\right] \cdot M_{robot}}{R_L^2}$$

$$F = \frac{6.67 \times 10^{-11}\left(7 \times 10^{24} + 8.14 \times 10^{22}\right) \times 65}{(7.7 \times 10^6)^2}$$

$$\boxed{F = 517\,N}$$

If they solve using planet density instead, they should have

$$F = \frac{G M_{Lander}}{R_L^2} \times M_{total} \quad \text{where}$$

$$M_{total} = \frac{4}{3}\pi R_{planet}^3 \rho_{planet} + \frac{4}{3}\pi\left(R_{Lander}^3 - R_{planet}^3\right)\rho_{dust}$$

So now the force becomes

$$F = \frac{4\pi G M_{Lander}}{3 R_L^2}\left(R_p^3 \rho_p + R_L^3 \rho_{dust} - R_p^3 \rho_{dust}\right)$$

where ρ_p = planet density, $\rho = \frac{7 \times 10^{24}}{\frac{4}{3}\pi(6.4 \times 10^6)^3} = 6375\ kg\ m^{-3}$

and $\boxed{F = 517\,N}$ again.

(b) Since $517\,N \ll 10^6\,N$, it survives the impact

Problem 2: Classical Mechanics

For people on Earth, it is practical to construct a right-handed coordinate system where $\hat{z}$ points up, away from the center of the earth. Then, $\hat{x}$ and $\hat{y}$ are perpendicular to $\hat{z}$. Let $\hat{y}$ point tangent to the Earth's surface and toward the East. The remaining component, $\hat{x}$, has to point tangent to the Earth's surface and somewhat to the south.

A particle was released from an extremely tall tower at a height, h, above the Earth's surface at a latitude of θ and longitude of ϕ. The object then fell to Earth. Ignoring air resistance and using any of the following: the Earth's radius (R), the Earth's rotational period (τ), the gravitational acceleration felt on and near the Earth (g), the latitude of the tower where the object was dropped (θ), the longitude of the tower where the object was dropped (ϕ), the mass of the object (m), the amount of time since the object was released (t), and the $\hat{x}$, $\hat{y}$, and $\hat{z}$ directions described above, what are the strength and direction of the Coriolis acceleration on the object due only to its fall toward the Earth?

Solution:

Let the angular velocity vector of the Earth's rotation = $\vec{\omega}$, which points north, parallel to the Earth's rotation axis. You can easily determine the magnitude of the $\vec{\omega}$ from the Earth's rotation period. Let us call it ω.

Let the velocity vector of the object's fall toward Earth be $\vec{v}_r$, which points in the $-\hat{z}$ direction. Note, we will find that the Coriolis acceleration points in the $\hat{y}$-direction. Technically, this adds a time-dependent velocity component in the $\hat{y}$-direction, but the phrasing of the question lets us neglect it.

Call the Coriolis acceleration $\vec{a}_c$. The general equation for the Coriolis acceleration is

$\vec{a}_c = -2\vec{\omega} \times \vec{v}_r$, where " $\times$" is the cross product. In order to evaluate the cross product, we need to break the $\vec{\omega}$ into components along the $\hat{x}$, $\hat{y}$, and $\hat{z}$ directions. The components are:

$$\omega_x = -\omega \cos\theta$$

$$\omega_y = 0$$

$$\omega_z = \omega \sin\theta$$

As for v_r, it is equal to gt directed in the $-\hat{z}$ direction. Carrying out the cross product results in:

$$\vec{\omega} \times \vec{v}_r = \hat{y}(-\omega_x v_{rz})$$

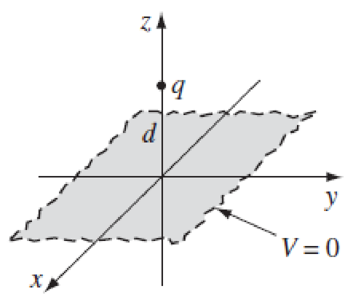
which simplifies to $\vec{\omega} \times \vec{v}_r = \hat{y}(\omega\cos\theta)(-gt)$, which can be rewritten as $\vec{\omega} \times \vec{v}_r = -\omega gt \cos\theta\, \hat{y}$. Using $\vec{a}_c = -2\vec{\omega} \times \vec{v}_r$ yields:

$$\vec{a}_c == 2\omega gt \cos\theta\, \hat{y}$$

Problem 3: Electromagnetism

Suppose a point charge q is held a distance d above an infinite grounded conducting plane (see figure below). Questions:

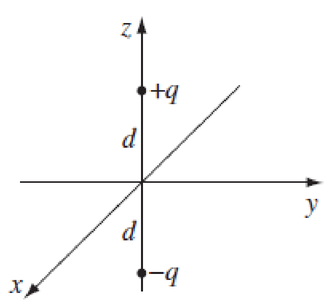
1) What is the potential in the region above the plane?
2) What is the induced surface charge density on the conducting plate? And what is the total induced charge?
3) What is the electrostatic force acting on charge q?
4) What is the electrostatic energy stored in the system?



Solution: We can use image charge method solve the problem. Due to the boundary condition,

1. $V = 0$ when $z = 0$,
2. $V \to 0$ for $x^2 + y^2 + z^2 \gg d^2$,

and the symmetry, the system can be treated as two point-charge system, one charge q at (0, 0, d) and the other charge -q at (0, 0, -d), as shown in the figure below:



1) For an arbitrary location Q at (x, y, z) at $z \geq 0$, the potential is given by Coulomb's law,

$$V(x,y,z) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right].$$

2) The induced surface charge density is determined by the local electric field at the conducting plane,

$$\sigma = -\varepsilon_0 \frac{\partial V}{\partial n},$$

where $\frac{\partial V}{\partial n}$ is the normal derivative of $V(x, y, z)$ at the surface. According to our configuration, $\frac{\partial V}{\partial n} = \frac{\partial V}{\partial z}|_{z=0}$. Here,

$$\frac{\partial V}{\partial z} = \frac{1}{4\pi\varepsilon_0}\left\{\frac{-q(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} + \frac{q(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}}\right\}.$$

Thus,

$$\sigma(x, y) = \frac{-qd}{2\pi(x^2+y^2+d^2)^{3/2}}$$

The total induced charge,

$$Q_{ind} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sigma(x, y)dxdy$$

Use the polar coordinates (r, φ), with $r^2 = x^2 + y^2$, $dxdy = rdrd\varphi$, and $\sigma(r) = \frac{-qd}{2\pi(r^2+d^2)^{3/2}}$, thus

$$Q_{ind} = \int_0^{2\pi}\int_0^{\infty}\frac{-qd}{2\pi(r^2+d^2)^{3/2}}rdrd\varphi = \frac{qd}{\sqrt{r^2+d^2}}\bigg|_0^{\infty} = -q.$$

3) According to the two point charge figure, the electrostatic force acting on charge q is,

$$\vec{F} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2d)^2}\hat{z}$$

4) The electrostatic energy stored for the two point-charge system is $W' = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{2d}$.
However, for the point charge-plane system, since the electric field beneath the conduction plane ($z < 0$) is zero (screening effect), the energy W stored should be halved, i.e.,

$$W = \frac{1}{2}W' = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{4d}$$

Problem 4: Electromagnetism

A square loop and a triangular loop are placed near an infinite straight wire as shown in the figure. The loop and the wire carry the same steady current *I*.

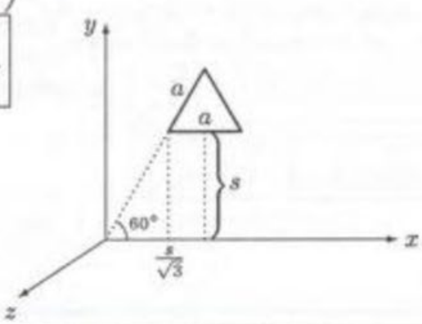
(a) Find the total magnetic force acting on the square loop as shown in Fig. (a)
(b) Find the total magnetic force acting on the triangular loop as shown in Fig. (b)

(a) (b)

Problem 5.10

(a) The forces on the two sides cancel. At the bottom, $B = \frac{\mu_0 I}{2\pi s} \Rightarrow F = \left(\frac{\mu_0 I}{2\pi s}\right) Ia = \frac{\mu_0 I^2 a}{2\pi s}$ (up). At the top, $B = \frac{\mu_0 I}{2\pi(s+a)} \Rightarrow F = \frac{\mu_0 I^2 a}{2\pi(s+a)}$ (down). The net force is $\boxed{\frac{\mu_0 I^2 a^2}{2\pi s(s+a)}}$ (up).

(b) The force on the bottom is the same as before, $\mu_0 I^2/2\pi$ (up). On the left side, $\mathbf{B} = \frac{\mu_0 I}{2\pi y}\,\hat{\mathbf{z}}$; $d\mathbf{F} = I(d\mathbf{l} \times \mathbf{B}) = I(dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}) \times \left(\frac{\mu_0 I}{2\pi y}\,\hat{\mathbf{z}}\right) = \frac{\mu_0 I^2}{2\pi y}(-dx\,\hat{\mathbf{y}} + dy\,\hat{\mathbf{x}})$. But the x component cancels the corresponding term from the right side, and $F_y = -\frac{\mu_0 I^2}{2\pi}\int_{s/\sqrt{3}}^{(s/\sqrt{3}+a/2)} \frac{1}{y}\,dx$. Here $y = \sqrt{3}x$, so $F_y = -\frac{\mu_0 I^2}{2\sqrt{3}\pi}\ln\left(\frac{s/\sqrt{3}+a/2}{s/\sqrt{3}}\right) = -\frac{\mu_0 I^2}{2\sqrt{3}\pi}\ln\left(1+\frac{\sqrt{3}a}{2s}\right)$. The force on the right side is the same, so the net force on the triangle is $\boxed{\frac{\mu_0 I^2}{2\pi}\left[1 - \frac{2}{\sqrt{3}}\ln\left(1+\frac{\sqrt{3}a}{2s}\right)\right]}$.



$$\boxed{\frac{\mu_0 I^2}{2\pi}\left[a/s - \frac{2}{\sqrt{3}}\ln\left(1+\frac{\sqrt{3}a}{2s}\right)\right]}.$$