SOLUTION

The University of Georgia
Department of Physics and Astronomy

Prelim Exam
August 13, 2021

Part I (problems 1, 2, 3, and 4)
9:00 am – 1:00 pm

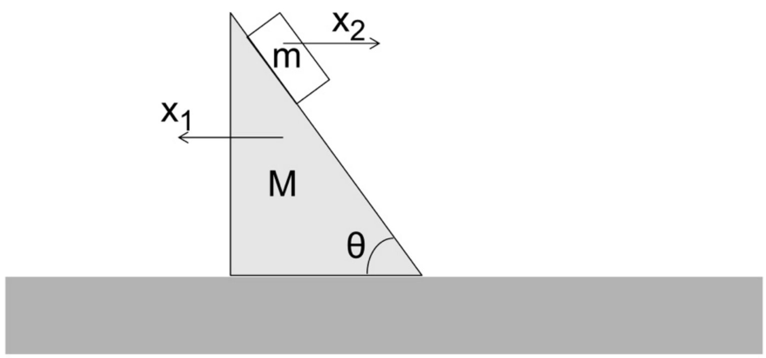
Instructions:

- Start each problem on a new sheet of paper. Write the problem number on the top left of each page and your pre-arranged prelim ID number (but ***not*** your name) on the top right of each page.
- Leave margins for stapling and photocopying.
- Write only on *one side* of the paper. Please ***do not*** write on the back side.
- If not advised otherwise, derive the mathematical solution for a problem from basic principles or general laws (Newton's laws, the Maxwell equations, the Schrödinger equation, etc.).
- You may use a calculator for basic operations only (i.e., not for referring to notes stored in memory, symbolic algebra, symbolic and numerical integration, etc.) The use of cell phones, tablets, and laptops is not permitted.
- Show your work and/or explain your reasoning in *all* problems, as the graders are not able to read minds. Even if your final answer is correct, not showing your work and reasoning will result in a *substantial* penalty.
- Write your work and reasoning in a neat, clear, and logical manner so that the grader can follow it. Lack of clarity is likely to result in a substantial penalty.

SOLUTION

Problem 1: Classical Mechanics

Initially, a block of mass m is held motionless on a frictionless wedge of mass M and angle of inclination θ (see diagram). The wedge rests on a frictionless horizontal surface. The block is released from rest. Choose variables such that x_1 is the position of the wedge to the *left* of some (arbitrary) origin and x_2 is the position of the block to the *right* of that same origin; i.e., $x_1 + x_2$ increases as the block slides down the wedge. [*Hint*: The wedge moves at the same time that the block moves, so the horizontal displacement of the block relative to the wedge is $x_1 + x_2$, not merely x_1.]



(a) How is the Lagrangian, $L(q_1, \ldots, q_N, \dot{q}_1, \ldots, \dot{q}_N)$, for a system defined? What is the general form of the Euler-Lagrange equation of motion for the ith degree of freedom, q_i?

(b) Write down the Lagrangian for this system in terms of variables x_1 and x_2, their first time derivatives $\dot{x}_1$ and $\dot{x}_2$, and constants g, M, m, and $\tan\theta$. [*Note*: Do not invent alternate variables or constants.]

(c) Use the Euler-Lagrange formalism to deduce equations of motion for x_1 and x_2. You do *not* have to solve either equation.

Solution:

(a) The Lagrangian is given by $L(q_i, \dot{q}_i) = T(q_i, \dot{q}_i) - V(q_i)$. The Euler-Lagrange equation of motion for the ith degree of freedom is:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}.$$

(b) For this problem, $y_1 = \text{constant} = 0$ and $y_2 = (x_1 + x_2)\tan\theta$, so that $\dot{y}_1 = 0$ and $\dot{y}_2 = (\dot{x}_1 + \dot{x}_2)\tan\theta$. Hence:

$$\begin{aligned} T &= \tfrac{1}{2}M(\dot{x}_1^2 + \dot{y}_1^2) + \tfrac{1}{2}m(\dot{x}_2^2 + \dot{y}_2^2) \\ &= \tfrac{1}{2}M\dot{x}_1^2 + \tfrac{1}{2}m[\dot{x}_2^2 + (\dot{x}_1 + \dot{x}_2)^2 \tan^2\theta]; \end{aligned}$$

$$V = mgy_2 = mg(x_1 + x_2)\tan\theta.$$

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The Lagrangian for this system is:

$$L = \tfrac{1}{2}M\dot{x}_1^2 + \tfrac{1}{2}m[\dot{x}_2^2 + (\dot{x}_1 + \dot{x}_2)^2 \tan^2\theta] - mg(x_1 + x_2)\tan\theta.$$

(c) The Euler-Lagrange equation applied to degree of freedom $i = 1$ yields:

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} &= \frac{d}{dt}[M\dot{x}_1 + m(\dot{x}_1 + \dot{x}_2)\tan^2\theta] \\ &= M\ddot{x}_1 + m(\ddot{x}_1 + \ddot{x}_2)\tan^2\theta\,;\end{aligned}$$

$$\frac{\partial L}{\partial x_1} = mg\tan\theta\,;$$

$$\therefore\quad \boxed{M\ddot{x}_1 + m(\ddot{x}_1 + \ddot{x}_2)\tan^2\theta = mg\tan\theta}\,.$$

The Euler-Lagrange equation applied to degree of freedom $i = 2$ yields:

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} &= \frac{d}{dt}[m\dot{x}_2 + m(\dot{x}_1 + \dot{x}_2)\tan^2\theta] \\ &= m\ddot{x}_2 + m(\ddot{x}_1 + \ddot{x}_2)\tan^2\theta\,;\end{aligned}$$

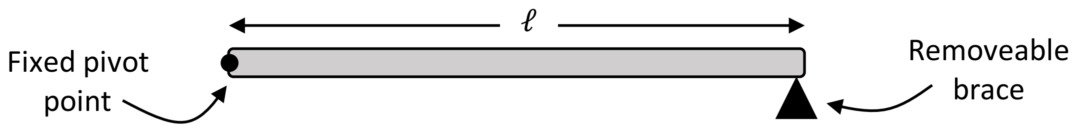
$$\frac{\partial L}{\partial x_2} = mg\tan\theta\,;$$

$$\therefore\quad \boxed{m\ddot{x}_2 + m(\ddot{x}_1 + \ddot{x}_2)\tan^2\theta = mg\tan\theta}\,.$$

SOLUTION

Problem 2: Classical Mechanics

Consider a uniform, thin rod of length l and mass m pivoted about one end. Initially the rod is oriented horizontally and held at rest by a removable brace. The brace is then removed, and the rod begins to fall due to gravity.



(a) Calculate the moment of inertia about the pivot. Find the distance from the pivot to the point at which, if all the mass were concentrated there, the moment of inertia about the pivot axis would be the same as the real moment of inertia. This distance is called the *radius of gyration*.

(b) What is the acceleration of the center-of-mass of the rod the instant after it is released? [*Hint*: The answer is *not* g.]

(c) What force does the pivot exert on the rod the instant after it is released?

Solution:

(a) The moment of inertia about the pivot is:

$$I = \int r^2 dm = \int_0^\ell x^2 \cdot \frac{m}{\ell} dx = \frac{m}{\ell}\frac{x^3}{3}\bigg|_0^\ell = \boxed{\tfrac{1}{3}m\ell^2}.$$

The radius of gyration, k, is:

$$mk^2 = \tfrac{1}{3}m\ell^2 \quad \Longrightarrow \quad \boxed{k = \tfrac{1}{\sqrt{3}}\ell}.$$

(b) The center-of-mass of the rod is at $r_{\text{cm}} = \ell/2$. If the angular acceleration about the pivot is α, then the acceleration of the center-of-mass is $a_{\text{cm}} = \alpha r_{\text{cm}} = \alpha\ell/2$. The angular acceleration is related to the torque and the moment of inertia by:

$$\tfrac{1}{3}m\ell^2\alpha = I\alpha = \tau_{\text{tot}} = \tfrac{1}{2}\ell mg \quad \Longrightarrow \quad \alpha = \frac{3g}{2\ell}.$$

Hence, the *downward* acceleration of the center-of-mass is:

$$a_{\text{cm}} = \alpha\ell/2 = \boxed{\tfrac{3}{4}g}.$$

(c) The pivot exerts an upward force F, and gravity exerts a downward force mg. Hence, by Newton's 2nd Law:

$$ma = mg - F \quad \Longrightarrow \quad F = mg - ma = mg - \tfrac{3}{4}mg \quad \Longrightarrow \quad F = \tfrac{1}{4}mg.$$

SOLUTION

Problem 3: Electromagnetism

A metal sphere of radius R, carrying a charge q, is surrounded by a thick concentric metal shell (inner radius a, outer radius b). The shell carries no net charge.

(a) Find the surface charge density at R, at a, and at b.

(b) Find the potential at the center, using infinity as the reference point.

(c) Find the capacitance of this system.

Solution:

(a) Since the sphere and the shell are both conductors, any net charge must reside at their surface(s). Hence, the surface charge density of the sphere is:

$$\sigma_R = \frac{q}{4\pi R^2}.$$

In order for the electric field inside the shell to vanish, a total charge $-q$ is induced on its inner surface, meaning a total charge q is induced on its outer surface (i.e., the *net* charge on the shell is zero). Hence:

$$\sigma_a = \frac{-q}{4\pi a^2} \quad \text{and} \quad \sigma_b = \frac{q}{4\pi b^2}.$$

(b) At $r = \infty$ the potential is zero, so at the center, $r = 0$, the potential is:

$$V(0) = -\int_\infty^0 \mathbf{E}\cdot d\boldsymbol{\ell} = -\int_\infty^b E_1 dr - \int_b^a E_2 dr - \int_a^R E_3 dr - \int_R^0 E_4 dr,$$

where:

$$E_1 = E_3 = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \quad \text{and} \quad E_2 = E_4 = 0.$$

Thus:

$$V(0) = -\int_\infty^b \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}dr - \int_a^R \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}dr = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{b} - \frac{1}{a} + \frac{1}{R}\right].$$

(c) There are two inequivalent ways of interpreting what is meant by "the capacitance" here. (i) On the one hand, you could think of this system as having two conductors, so the capacitance is the charge stored on them ($+q$ on one, $-q$ on the other) per unit potential difference between them, $C = q/[V(R) - V(a)]$. This is more aligned with the definition one would find in, say, Griffiths. (ii) On the other hand, you could think of the spherical shell conductor as being like an infinite dielectric between the spherical conductor charged to $+q$ and the putative

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"conductor at infinity" with charge $-q$. Then $C = q/[V(0) - V(\infty)]$. This interpretation is what one might guess from the parts that came before, describing how the system was charged. Given the ambiguity, full credit was awarded for either solution. Below are both possibilities:

(i) Imagine charge $+q$ on the inner (spherical) conductor. Then the potential difference between the inner conductor (sphere) and outer conductor (shell) is:

$$\Delta V = -\int_a^R \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,dr = \frac{1}{4\pi\epsilon_0}\frac{q}{r}\bigg|_a^R = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{R} - \frac{1}{a}\right] = \frac{q}{4\pi\epsilon_0}\frac{a-R}{aR}.$$

Hence, the capacitance is given by:

$$C = \frac{q}{\Delta V} = \frac{4\pi\epsilon_0 aR}{a-R}.$$

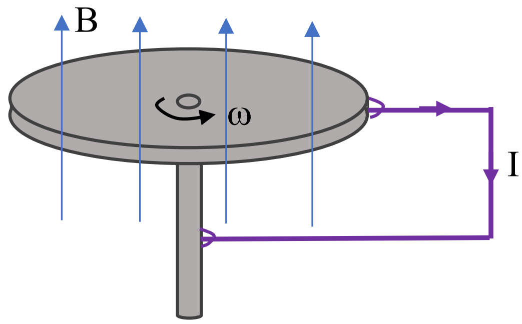
(ii) Here $V(0) - V(\infty)$ the same as was calculated in part (b). Therefore, for this interpretation, the capacitance is given by:

$$C = \frac{q}{\Delta V} = \frac{q}{V(0) - V(\infty)} = 4\pi\epsilon_0\left[\frac{1}{b} - \frac{1}{a} + \frac{1}{R}\right]^{-1} = \frac{4\pi\epsilon_0 \cdot abR}{aR - bR + ab}.$$

SOLUTION

Problem 4: Electromagnetism

Faraday disk generator: As shown in the figure below, consider a copper disk spinning in a uniform external magnetic field $\vec{B} = B\hat{z}$. Let a be the radius of the disk, and let $\vec{\omega} = \omega\hat{z}$ be its angular velocity. There is a conducting wire fixed in space, with sliding contacts at the rim and the center of the disk. The total resistance of the system is R.



(a) Find the current I flowing through the wire.

(b) Find the torque τ generated by the rotating disk.

Solution:

(a) The EMF of the rotating disk can be expressed as:

$$\varepsilon = \oint_C \vec{v} \times \vec{B} \cdot d\vec{l},$$

so that:

$$\varepsilon = \int_0^a \omega r B dr = \frac{1}{2} B\omega a^2.$$

Equivalently, if one considers any free charge q in the disk at the radius r from the center, under the uniform magnetic field $\vec{B}$, this charge will feel a Lorentzian force:

$$\vec{F} = q\vec{v} \times \vec{B} = q\omega r B\hat{r}.$$

Thus, there is an effective electric field generated along the radial direction:

$$\vec{E} = \frac{\vec{F}}{q} = \omega r B\hat{r}.$$

Hence, the generated EMF is (once again):

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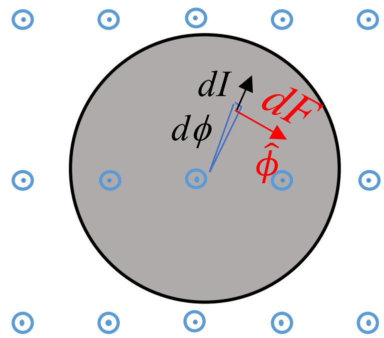
$$\varepsilon = \int_0^a \omega r B dr = \frac{1}{2} B\omega a^2.$$

In any event, according to the Ohm's law, the current flowing through wire is:

$$I = \frac{\varepsilon}{R} = \frac{1}{2R} B\omega a^2.$$

(b) To calculate the torque, assume that the linear current density $\vec{J}(r)$ in an area element $dA = rtd\phi$, where r is the radius and t is the thickness of the disk, and the magnetic force $d\vec{F}$ can be expressed as:



$$d\vec{F} = J(r)rtd\phi B\hat{\phi}.$$

In addition, according to the conservation of charge or current, one has:

$$I = \oint J(r)rtd\phi = \int_o^{2\pi} J(r)rtd\phi = 2\pi J(r)rt \quad \Longrightarrow \quad J(r) = \frac{I}{2\pi rt}.$$

Thus, $d\vec{F} = \frac{IB}{2\pi} d\phi dr\hat{\phi}$, and $d\vec{\tau} = \vec{r} \times d\vec{F} = -r\frac{IB}{2\pi} d\phi dr\hat{z}$. Hence:

$$\vec{\tau} = -\frac{IB}{2\pi}\hat{z}\int_0^a dr \int_0^{2\pi} rd\phi = -\frac{IBa^2}{2}\hat{z}.$$

SOLUTION

The University of Georgia
Department of Physics and Astronomy

Prelim Exam
August 13, 2021

Part II (problems 5 and 6)
2:00 pm – 4:00 pm

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SOLUTION

Problem 5: Quantum Mechanics

A particle of mass m in a 1D infinite square well of width a has as its initial wave function an even mixture of the first two energy eigenstates:

$$\psi(x,0) = A[\varphi_1(x) + \varphi_2(x)].$$

(a) Find A that normalizes $\psi(x,0)$.

(b) Find $\psi(x,t)$ and $|\psi(x,t)|^2$. Express the latter as a sinusoidal function of time. [*Note*: To simplify the result, use $\omega \equiv \pi^2\hbar/2ma^2$.]

(c) Calculate the expectation value of position, $\langle x(t)\rangle$. Notice that it oscillates in time. What is the angular frequency of oscillation? What is the amplitude of oscillation?

(d) If you measured the energy of this particle, what values might you get? What is the probability of getting each of them? Calculate the expectation value of energy. How does it compare with E_1 and E_2, the two lowest energy eigenvalues?

[For the 1D infinite square well, $\varphi_n(x) = \sqrt{2/a}\,\sin(n\pi x/a)$ and $E_n = n^2\pi^2\hbar^2/2ma^2$.]

Solution:

(a) Normalizing the wavefunction gives A:

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} |\psi(x,0)|^2 dx \\ &= |A|^2\left[\int_{-\infty}^{\infty}|\varphi_1|^2 dx + \int_{-\infty}^{\infty}|\varphi_2|^2 dx + \int_{-\infty}^{\infty}\varphi_1^*\varphi_2 dx + \int_{-\infty}^{\infty}\varphi_2^*\varphi_1 dx\right]. \end{aligned}$$

The first and second terms are both 1 and the third and fourth terms are both 0, because the $\{\varphi_i\}$ are orthonormal. Hence:

$$1 = |A|^2[1+1+0+0] \quad\Longrightarrow\quad |A|^2 = 1/2.$$

The overall phase is arbitrary, so we choose, by convention, to have A be real. That means $A = 1/\sqrt{2}$.

(b) Since $\psi(x,0)$ is given as an expansion in energy eigenstates, all we have to do to obtain $\psi(x,t)$ is multiply each term by the phase factor $e^{-iE_nt/\hbar} \equiv e^{-in^2\omega t}$:

$$\psi(x,t) = \frac{1}{\sqrt{a}}\left[\sin\left(\frac{\pi x}{a}\right)e^{-i\omega t} + \sin\left(\frac{2\pi x}{a}\right)e^{-4i\omega t}\right].$$

The magnitude squared of the above is:

$$|\psi(x,t)|^2 = \frac{1}{a}\left[\sin^2\left(\frac{\pi x}{a}\right) + \sin^2\left(\frac{2\pi x}{a}\right) + 2\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\cos(3\omega t)\right].$$

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(c) The expectation value of position is given by:

$$
\begin{aligned}
\langle x(t) \rangle &= \int x|\psi(x,t)|^2 dx \\
&= \frac{1}{a}\int_0^a x\sin^2\left(\frac{\pi x}{a}\right) dx + \frac{1}{a}\int_0^a x\sin^2\left(\frac{2\pi x}{a}\right) dx \\
&\qquad + \frac{2}{a}\cos(3\omega t)\int_0^a x\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right) dx \\
&= \frac{a}{2} - \frac{16a}{9\pi^2}\cos(3\omega t).
\end{aligned}
$$

The angular frequency of oscillation is 3ω, and the amplitude of oscillation is $16a/9\pi^2$.

(d) Given that the initial state is an even mixture of the two lowest energy eigenstates, the only values of energy that can be measured are $E_1 = \hbar\omega$ and $E_2 = 4\hbar\omega$. Moreover, the probability of measuring each energy value is $\mathcal{P}_1 = \mathcal{P}_2 = 1/2$. Since energy is conserved, these probabilities do not vary with time. The expectation value of energy is computed as follows:

$$
\langle E \rangle = \sum_n E_n \mathcal{P}_n = \tfrac{1}{2}E_1 + \tfrac{1}{2}E_2 = \frac{\hbar\omega + 4\hbar\omega}{2} = \tfrac{5}{2}\hbar\omega.
$$

Clearly, the $\langle E \rangle$ is halfway between E_1 and E_2.

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Problem 6: Quantum Mechanics

Consider the quantum subspace corresponding to total angular momentum $\ell = 2$. For this subspace, what are the eigenvalues of the following operators?

(a) $\hat{L}_z$

(b) $\frac{3}{5}\hat{L}_x - \frac{4}{5}\hat{L}_y$

(c) $2\hat{L}_x - 6\hat{L}_y + 3\hat{L}_z$

Solution:

(a) Obviously, the eigenvalues, in units of $\hbar$, of the $\hat{L}_z$ operator for the $\ell = 2$ subspace are $+2, +1, 0, -1, -2$.

(b) The operator $\frac{3}{5}\hat{L}_x - \frac{4}{5}\hat{L}_y$ can be regarded as the dot product $\mathbf{n} \cdot \hat{\mathbf{L}}$, where the vector $\mathbf{n} = \frac{3}{5}\hat{e}_x - \frac{4}{5}\hat{e}_y$ and $\hat{\mathbf{L}}$ is the angular momentum vector operator. Thus $\mathbf{n}$ is a unit vector. If we rotate the frame so that the new z axis coincides with the $\mathbf{n}$ direction, we recover the $\hat{L}_z$ operator for the new frame. Because of rotational invariance, the eigenvalues are unchanged. Hence, the eigenvalues of the operator $\frac{3}{5}\hat{L}_x - \frac{4}{5}\hat{L}_y$, in units of $\hbar$, are $+2, +1, 0, -1, -2$.

(c) The operator $2\hat{L}_x - 6\hat{L}_y + 3\hat{L}_z$ can also be represented as a dot product, but now with a vector $\mathbf{n}$ of magnitude 7. Using the same argument as in (b), we find the eigenvalues of this operator, in units of $\hbar$, are $+14, +7, 0, -7, -14$.