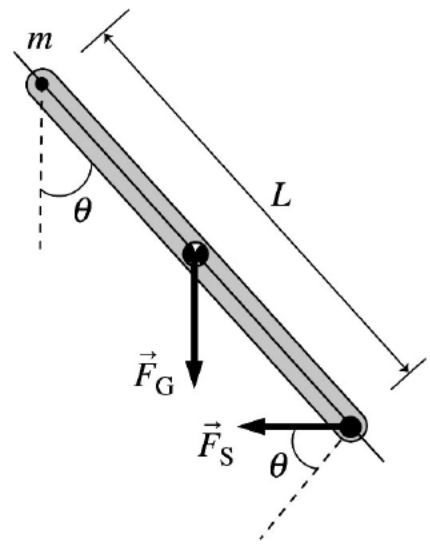
Solutions to January Prelim Exam Problems

1.



Solve: We must derive our own equation for this combination of a pendulum and spring. For small oscillations, $\vec{F}_s$ remains horizontal. The net torque around the pivot point is

$$\tau_{\text{net}} = I\alpha = -F_s L\cos\theta - F_G\left(\frac{L}{2}\right)\sin\theta$$

With $\alpha = \dfrac{d^2\theta}{dt^2}$, $F_G = mg$, $F_s = k\Delta x = kL\sin\theta$, and $I = \dfrac{1}{3}mL^2$,

$$\frac{d^2\theta}{dt^2} = -\frac{3k}{m}\sin\theta\cos\theta - \frac{3g}{2L}\sin\theta$$

We can use $\sin\theta\cos\theta = \dfrac{1}{2}\sin 2\theta$. For small angles, $\sin\theta \approx \theta$ and $\sin 2\theta \approx 2\theta$. So

$$\frac{d^2\theta}{dt^2} = -\left(\frac{3k}{m} + \frac{3g}{2L}\right)\theta$$

This is the same as Equations 15.32 and 15.46 with

$$\omega = \sqrt{\frac{3k}{m} + \frac{3g}{2L}}$$

The frequency of oscillation is thus

$$f = \frac{1}{2\pi}\sqrt{\frac{3(3.0\text{ N/m})}{(0.200\text{ kg})} + \frac{3(9.8\text{ m/s}^2)}{2(0.20\text{ m})}} = 1.73\text{ Hz}$$

The period $T = \dfrac{1}{f} = 0.58$ s.

2.

(1) Let's consider accelerations at the equator. At the maximum rotation speed, gravity and centrifugal force are equal in magnitude.

$$(2\pi R/T)^2/_R = {G(4\pi R^3\rho/3)}/_{R^2}$$

Solve for T and you can get

$$T = \sqrt{3\pi/G\rho}$$

(2) At latitude φ,

$$(2\pi r/T)^2/_r = {G(4\pi R^3\rho/3)}/_{R^2}\cos(\varphi)$$

$$4\pi^2 r/T^2 = G(4\pi R\rho/3)\cos(\varphi),\ r = R\cos(\varphi)$$

$$4\pi^2 R\cos(\varphi)/T^2 = G(4\pi R\rho/3)\cos(\varphi)$$

$$\pi/T^2 = G\,\rho/3$$

$$T = \sqrt{3\pi/G\rho}$$

(3) For the Earth, $\rho = 5500 kg/m^3$ and $G = 6.67\times10^{-11} m^3/kg\ sec^2$.

$T_{min} \approx 5070\ sec$ = 84.5 minutes

3.

a) Assume that the solenoid is an ideal solenoid; that is

$$\overline{B} = \mu_0 N I \hat{k}$$

If the current in the solenoid increases, the strength of the magnetic field also increases. The rate of change in the strength of the magnetic field is equal to

$$\frac{d\overline{B}}{dt} = \mu_0 N \frac{dI}{dt}\hat{k} = \mu_0 N k\, \hat{k}$$

The magnetic flux intercepted by the wire loop is equal to

$$\Phi = \pi\, \mathrm{a}^2 B$$

The corresponding rate of change of the magnetic flux is equal to

$$\frac{d\Phi}{dt} = \pi\, a^2 \frac{dB}{dt} = \pi\, a^2 \mu_0 N k$$

The induced emf can be obtained from the flux law:

$$\varepsilon = -\frac{d\Phi}{dt} = -\pi\, a^2 \mu_0 N k$$

The current induced in the wire loop is equal to

$$I = \frac{\varepsilon}{R} = \frac{\pi\, a^2}{R}\mu_0 N k$$

The solenoidal magnetic field points from left to right. An increase in the strength of the magnetic field will induce a current in the loop directed such that the magnetic field it produces point from right to left (Lenz's law). Therefore, the current flows from left to right through the resistor.

b) The change in the magnetic flux enclosed by the wire loop is equal to

$$\Delta\Phi = 2\pi a^2 \mu_0 N I$$

The current flowing through the resistor is equal to

$$I = \frac{\varepsilon}{R} = \frac{1}{R}\frac{d\Phi}{dt} = \frac{dQ}{dt}$$

This relation shows that

$$\Delta Q = \int_{-\infty}^{\infty} \frac{dQ}{dt} dt = -\frac{1}{R} \int_{-\infty}^{\infty} \frac{d\Phi}{dt} dt = -\frac{\Delta\Phi}{R}$$

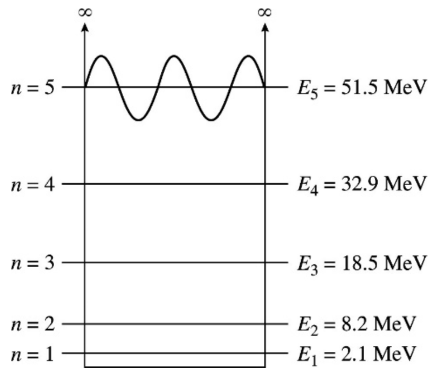
Substituting the expression for DF we obtain

$$\left| \Delta Q \right| = \frac{1}{R}\left(2\pi a^2 \mu_0 N I\right)$$

4.

included separately

5.



Solve: (a) The energy levels are $E_n = n^2h^2/8mL^2$. Two adjacent energy levels have the energy ratio

$$\frac{E_{n+1}}{E_n} = \frac{(n+1)^2}{n^2} \Rightarrow \frac{n+1}{n} = \sqrt{\frac{E_{n+1}}{E_n}} = \sqrt{\frac{51.4\text{ MeV}}{32.9\text{ MeV}}} = 1.25 = \frac{5}{4} \Rightarrow n = 4 \text{ and } n+1 = 5$$

(b) We have $E_n = n^2E_1$, so $E_1 = 32.9\text{ MeV}/16 = 2.06\text{ MeV}$. We can then find $E_2 = 8.2\text{ MeV}$ and $E_3 = 18.5\text{ MeV}$.

(c) ψ_5 has five antinodes and is zero at $x = 0$ fm and $x = L$.

(d) The photon energy is $E_{\text{photon}} = hf = hc/\lambda = \Delta E_{nm}$. Hence,

$$\lambda = \frac{hc}{\Delta E} = \frac{(4.14\times10^{-15}\text{ eV s})(3.00\times10^8\text{ m/s})}{51.4\times10^6\text{ eV} - 32.9\times10^6\text{ eV}} = 6.71\times10^{-5}\text{ nm}$$

This is a factor 10^7 smaller than typical visible-light wavelengths.

(e) Using $E_4 = 32.9\text{ MeV} = 5.26\times10^{-12}$ J,

$$m = \frac{4^2h^2}{8E_4L^2} = \frac{2(6.63\times10^{-34}\text{ J s})^2}{(5.26\times10^{-12}\text{ J})(10\times10^{-15}\text{ m})^2} = 1.67\times10^{-27}\text{ kg}$$

This is either a proton or a neutron.

6.

(a) The energy eigenvalue equation is:

$$-\frac{\hbar^2}{2mr_0^2}\frac{d^2\psi(\phi)}{d\phi^2} = E\psi(\phi) \quad \Longrightarrow \quad \frac{d^2\psi}{d\phi^2} = -\frac{2mr_0^2 E}{\hbar^2}\psi \equiv -k^2\psi,$$

which has solutions:

$$\psi(\phi) = Ae^{\pm ik\phi},$$

with corresponding energy $E = \hbar^2 k^2/2mr_0^2$. The normalization condition is:

$$1 = \int_0^{2\pi} \psi^*(\phi)\psi(\phi)d\phi = |A|^2 \int_0^{2\pi} d\phi = 2\pi|A|^2 \quad \Longrightarrow \quad A = \frac{1}{\sqrt{2\pi}}.$$

Continuity of the wave function requires $\psi(\phi + 2\pi) = \psi(\phi)$:

$$Ae^{\pm ik(\phi+2\pi)} = Ae^{\pm ik\phi} \quad \Longrightarrow \quad e^{\pm 2\pi k i} = 1 \quad \Longrightarrow \quad k = n \in \mathbb{Z}\,.$$

Hence, the normalized energy eigenfunctions are:

$$\psi_n(\phi) = \frac{1}{\sqrt{2\pi}}e^{in\phi},$$

and the corresponding energy eigenvalues are:

$$E_n = \frac{\hbar^2 n^2}{2mr_0^2},$$

where $n \in \mathbb{Z}$. Note that the ground-state energy, E_0, is nondegenerate, but that all other energies, $E_{n\neq 0}$, are two-fold degenerate, belonging to both ψ_n and ψ_{-n}. We explain this physically by noting that ψ_n and ψ_{-n} are waves propagating counter-clockwise and clockwise, respectively, about the ring. Given that $V(\phi) = 0$, there is nothing to distinguish the sense of propagation.

(b) The matrix element of $\mathcal{H}'$ for the unperturbed state ψ_n is zero:

$$\langle\psi_n|\mathcal{H}'|\psi_n\rangle = \frac{\epsilon}{2\pi}\int_0^{2\pi} \phi(\phi-\pi)(\phi-2\pi)\, d\phi = 0,$$

because the integrand is antisymmetric about $\phi = \pi$. For the (nondegenerate) ground state, $n = 0$, this matrix element *is* the first-order energy correction. So we need to go to second order:

$$E_0^{(2)} = \sum_{k\neq 0} \frac{|\langle\psi_0|\mathcal{H}'|\psi_k\rangle|^2}{E_0^{(0)} - E_k^{(0)}} = -\frac{2mr_0^2}{\hbar^2}\sum_{k\neq 0}\frac{|\langle\psi_0|\mathcal{H}'|\psi_k\rangle|^2}{k^2}.$$

The off-diagonal matrix elements are:

$$\langle\psi_0|\mathcal{H}'|\psi_k\rangle = \frac{\epsilon}{2\pi}\int_0^{2\pi} \phi(\phi-\pi)(\phi-2\pi)e^{ik\phi}\, d\phi = \frac{6\epsilon i}{k^3},$$

so that the second-order energy correction is:

$$E_0^{(2)} = -\frac{72mr_0^2\epsilon^2}{\hbar^2}\sum_{k=1}^{\infty}\frac{1}{k^8} = -\frac{72mr_0^2\epsilon^2}{\hbar^2}\frac{\pi^8}{9450} = -\frac{4\pi^8 mr_0^2\epsilon^2}{525\hbar^2}.$$

For the $n \neq 0$ states, we have to use degenerate perturbation theory, which means we need to compute the matrix elements of the perturbation Hamiltonian in the 2×2 degenerate subspace spanned by the states $|n\rangle$ and $|-n\rangle$. We already showed that the diagonal matrix elements are *zero*. The off-diagonal matrix elements are:

$$\begin{aligned}\langle\psi_{-n}|\mathcal{H}'|\psi_n\rangle &= \frac{\epsilon}{2\pi}\int_0^{2\pi} e^{in\phi}\phi(\phi-\pi)(\phi-2\pi)e^{in\phi}\,d\phi \\ &= \frac{\epsilon}{2\pi}\int_0^{2\pi}\phi(\phi-\pi)(\phi-2\pi)e^{i2n\phi}\,d\phi = \frac{3\epsilon i}{4n^3}\,, \end{aligned}$$

$$\langle\psi_n|\mathcal{H}'|\psi_{-n}\rangle = \langle\psi_{-n}|\mathcal{H}'|\psi_n\rangle^* = -\frac{3\epsilon i}{4n^3}.$$

Thus the perturbation submatrix takes the form:

$$\mathcal{H}'_{\text{sub}} \doteq \begin{bmatrix} 0 & -ic \\ ic & 0 \end{bmatrix},$$

where $c = 3V_0/4n^3$. This matrix has eigenvalues $\pm c$ and eigenvectors:

$$|v_+\rangle \doteq \begin{bmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix} \quad \text{and} \quad |v_-\rangle \doteq \begin{bmatrix} i/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

In other words:

$$E_{\pm n}^{(1)} = \pm\frac{3V_0}{4n^3},$$

and the appropriate combinations of unperturbed energy eigenstates that are also eigenstates of the perturbation are:

$$\psi_+(\phi) = \frac{1}{\sqrt{2}}\big(\psi_n(\phi) + i\psi_{-n}(\phi)\big)\,,$$

$$\psi_-(\phi) = \frac{1}{\sqrt{2}}\big(i\psi_n(\phi) + \psi_{-n}(\phi)\big)\,.$$