Physics Prelim Exam Fall 2014

Day 2:

Tues. August 12, 2014, 9:00am-1:00pm

Department of Physics and Astronomy

Problem 2.1: Modern Physics

Problem 2.2: Modern Physics

Problem 2.3: Quantum Mechanics

Problem 2.4: Quantum Mechanics

Problem 2.5: Thermal Physics

Problem 2.1: Modern Physics

Police radar detects the speed of a car as follows: Microwaves of a precisely known frequency are broadcast toward the car. The moving car reflects the microwaves with a Doppler shift. The reflected waves are received and combined with an attenuated version of the transmitted wave. Beats occur between the two microwave signals. The beat frequency is measured.

(a) For an electromagnetic wave reflected back to its source from a mirror approaching at a speed v, show that the reflected wave has frequency

$$f = f_{source}\frac{c+v}{c-v}$$

Where f_{source} is the source frequency.

(b) When v is much less than c, the beat frequency is much less than the transmitted frequency. In this case, use the approximation $f + f_{source} \cong 2f_{source}$, and show that the beat frequency can be written as $f_b = 2v/\lambda$

(c) If the source wavelength is 3.00 cm and the beat frequency measurement is accurate to $\pm 5Hz$, how accurate is the velocity measurement?

Solution 2.1:

(a) Let f_c be the frequency as seen by the car, thus $f_c = f_{source}\sqrt{\frac{c+v}{c-v}}$

Let f be the frequency of the reflected wave, thus $f = f_c\sqrt{\frac{c+v}{c-v}}$

Combining gives

$$f = f_{source}\frac{c+v}{c-v}$$

(b) Using the above result

$f(c-v) = f_{source}(c+v)$

$(f - f_{source})c = (f + f_{source})v \approx 2f_{source}v$

The beat frequency is then

$$f_b = f - f_{source} = \frac{2f_{source}v}{c} = \frac{2v}{\lambda}$$

(c) Using the above result,

$v = \frac{f_b\lambda}{2}$, so $\Delta v = \frac{\Delta f_b\lambda}{2} = \frac{(5Hz)(0.03m)}{2} = 0.075\,\frac{m}{s} \approx 0.2\ mi/h$

Problem 2.2: Modern Physics

When the Sun is directly overhead, the thermal energy incident on the Earth is 1.36 kW / m^2. Assuming that the Sun behaves like a perfect blackbody of radius $R_{sun} = 7\times10^5$ km and the Sun-Earth distance (d) is 1.5×10^8 km, (1) estimate the Sun's effective temperature, and (2) show that the Sun's luminosity (i.e., total energy emitted per unit time) is 3.8×10^{26} W/sec, and (3) find the temperature of the Earth that is in a radiative equilibrium with the Sun.

Constants:
Stefan-Boltzman constant $\sigma = 5.67 \times 10^{-8}\ Wm^{-2}K^{-4}sec^{-1}$

(1) The total intensity of radiation emitted on the surface the Sun per unit area is σT_{eff}^4, hence the total radiation from the entire surface of the Sun is $\sigma T_{eff}^4 \times 4\pi R_{sun}^2$. At the distance of the Earth, this radiation is diluted onto the surface of the sphere with a radius 1.5×10^8 km and the amount of radiation measured over one square meter is $1.36\ kW/m^2$. Therefore,

$$1.36\, kW/m^2 = \sigma T_{eff}^4 \times \left(\frac{R_{sun}^2}{d^2}\right)$$

solving for T_{eff},

$$\begin{aligned} T_{eff}^4 &= \frac{1360\, W/m^2}{\sigma}\left(\frac{1.5 \times 10^8}{7 \times 10^5}\right)^2 \\ &= \frac{1360}{5.67 \times 10^{-8}}\left(\frac{1500}{7}\right)^2 \\ T_{eff} &= 5,760\, K \end{aligned}$$

(2) The total radiation from the surface of the Sun is

$$\begin{aligned} L &= \sigma T_{eff}^4 \times surface\, are\, of\, the\, Sun \\ &= \sigma T_{eff}^4 \times 4\pi R_{sun}^2 \\ &= 5.67 \times 10^{-8}(5760)^4 \times 4\pi(7 \times 10^8) \\ &= 3.8 \times 10^{26}\, W/sec \end{aligned}$$

(3) For the Earth in a radiative equilibrim,

$$\begin{aligned} incident\, energy &= radiated\, energy \\ 1360\, kW/m^2 \times \pi R_{Earth}^2 &= 4\pi R_{Earth}^2 \sigma T_{Earth}^4 \\ T_{Earth} &= \left(\frac{1360}{4\sigma}\right)^{1/4} \\ &= 278\, K \end{aligned}$$

Problem 2.3: Quantum Mechanics

Compton realized in 1921 that if X-rays with frequency ν hit quasi-free crystal electrons, excited electrons do not only emit light of the same frequency. He also observed another signal with frequency $\nu' \leq \nu$. The experiments revealed further that the difference of the respective wavelengths $\Delta\lambda = c/\nu' - c/\nu$ (c: speed of light) depends only on the scattering angle, but is independent of the target material.

In 1923, Compton himself explained this effect on the basis of Einstein's hypothesis that photons can be considered as particles with energy $h\nu$, momentum $\boldsymbol{p}_\gamma$, and zero rest mass.

Write down the balance equations (relativistic problem!) for the total energy and momentum before ($\boldsymbol{p}_e = \boldsymbol{0}$) and after the collision. Determine $\Delta\lambda$ as a function of the angle θ between incident and emitted light. Calculate $\Delta\lambda$ specifically for $\theta = \pi/2$ ("Compton wavelength of electron").

Constants:

$h \approx 6.62607 \times 10^{-34}$ Js

$m_e \approx 9.10938 \times 10^{-31}$ kg

$c \approx 2.99792 \times 10^{8}$ m/s

Solution 2.3

Photon energy: $E_\gamma = h\nu = |\boldsymbol{p}_\gamma|c$, photon momentum: $\boldsymbol{p}_\gamma = \frac{h\nu}{c}\boldsymbol{n}$ with $\boldsymbol{n}$: unit vector of direction of photon propagation, electron energy: $E_e = \sqrt{m_e^2 c^4 + \boldsymbol{p}_e^2 c^2}$

<u>Energy balance:</u>

before collision: $E = E_\gamma + E_e = h\nu + m_e c^2$

after collision: $E' = E'_\gamma + E'_e = h\nu' + \sqrt{m_e^2 c^4 + \boldsymbol{p}_e'^2 c^2}$

balance: $E = E'$

<u>Momentum balance:</u>

before: $\boldsymbol{p} = \boldsymbol{p}_\gamma + \boldsymbol{p}_e = \frac{h\nu}{c}\boldsymbol{n}$

after: $\boldsymbol{p}' = \boldsymbol{p}'_\gamma + \boldsymbol{p}'_e = \frac{h\nu'}{c}\boldsymbol{n}' + \boldsymbol{p}'_e$

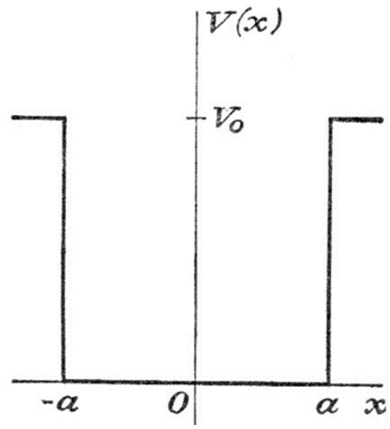
balance: $\boldsymbol{p} = \boldsymbol{p}'$

Thus, $\boldsymbol{p}'_e = \frac{h}{c}(\nu\boldsymbol{n} - \nu'\boldsymbol{n}')$. Substituting square $\boldsymbol{p}_e'^2 = \frac{h^2}{c^2}(\nu^2 + \nu'^2 - 2\nu\nu'\cos\theta)$, where θ is the angle between $\boldsymbol{n}$ and $\boldsymbol{n}'$, in energy balance yields $1/\nu' - 1/\nu = \frac{h}{m_e c^2}(1 - \cos\theta)$. Thus, $\Delta\lambda = \frac{h}{m_e c}(1 - \cos\theta)$.

For $\theta = \pi/2$, it is $\Delta\lambda = \frac{h}{m_e c} \approx 2.426 \times 10^{-12}$m.

Problem 2.4: Quantum Mechanics

Consider a quantum mechanical particle of mass m in a "square" one dimensional potential well shown to the right. For $-a \le x \le +a$ the potential $V(x) = 0$; otherwise, $V(x) = V_o$.



(a) Write down the Schroedinger equation for the particle.

(b) Suppose $V_o = \infty$:

i. What are the boundary conditions on the wave functions for the particle?

ii. What are the possible normalized wave functions for the particle?

iii. What are the possible energies of the particle?

iv. Draw the wave functions for the lowest three energy levels.

Solutions 2.4

a.)

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi\,, \qquad -a \le x \le +a$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_o\psi = E\psi\,, \quad otherwise$$

b) *i.)* $\psi(-a) = \psi(+a) = 0$

ii) $$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi\,, \qquad -a \le x \le +a$$

The general form of the solution is:

$\psi(x) = A\sin(\alpha x) + B\cos(\alpha x)$ where $\alpha = (2mE/\hbar^2)^{1/2}$

Considering the boundary conditions, we find two possible sets of solutions:

$A = 0$ and $\cos(\alpha a) = 0$ - (I.)

$B = 0$ and $\sin(\alpha a) = 0$ - (II.)

Thus, either $\alpha a = n\pi/2$ where n is an odd integer, for set (I.),

or

$\alpha a = n\pi/2$ where n is an even integer, for set (II.)

Hence,

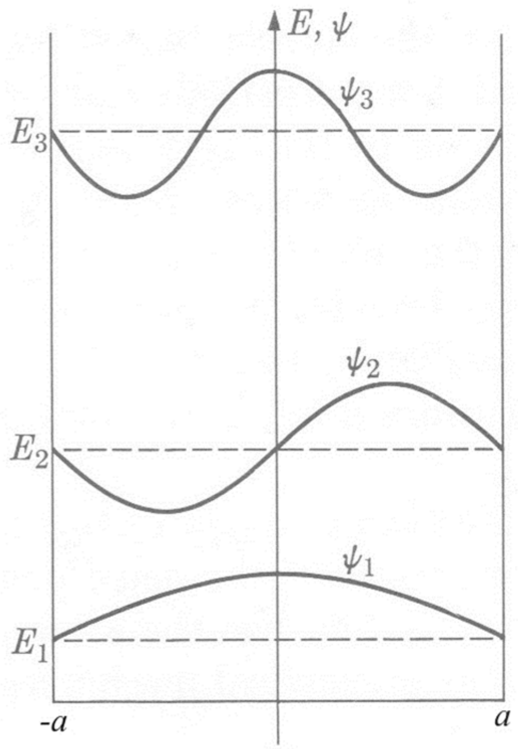
$$\psi(x) = B\cos\left(\frac{n\pi x}{2a}\right), \quad n\ odd$$

$$\psi(x) = A\sin\left(\frac{n\pi x}{2a}\right), \quad n\ even$$

If the wave functions are normalized, $A = B = 1$.

iii.) Inserting the wave functions into the Schroedinger eqn. we find: $E_n = \dfrac{\pi^2\hbar^2 n^2}{8ma^2}$

iv) Show solutions for n=1,2,3:


E, ψ
ψ3
E3
ψ2
E2
ψ1
E1
-a
a

Problem 2.5: Thermal Physics

Consider a gas with identical particles at temperature T and pressure p in a closed box with volume V. The internal energy is U and the entropy is S. There are no external forces or fields. Because the box is closed, no particle exchange with the environment is possible.

(a) State the first and second law of thermodynamics for this system, both in their differential forms.

(b) Show that in equilibrium the natural (independent) variables of U are S and V.

(c) Suppose the box is isolated (no heat and work exchange with the environment). Show that in this case the entropy can only increase and that it is maximal in equilibrium.

(d) Consider the situation that the box with constant volume is only mechanically, but not thermally isolated. The gas can exchange heat with the surrounding heat bath at constant temperature T. The natural variables are T and V. By performing an appropriate transformation of $U(S,V)$, exchange S by T. Show that the thus introduced thermodynamic potential (free energy) $F(T,V)$ can only decrease and that it is minimal in equilibrium.

Solution 2.5

The particle number is irrelevant in this problem, $dN = 0$.

(a) First law: $dU = \delta Q + \delta W$, where δQ is the heat and $\delta W = -pdV$ is the work,
Second law: $dS \geq \delta Q/T$ ($dS = \delta Q/T$ for reversible processes only).

(b) In equilibrium: $dU = TdS - pdV = \left(\frac{\partial U}{\partial S}\right)_{V,N} dS + \left(\frac{\partial U}{\partial V}\right)_{S,N} dV$, so that indeed $\left(\frac{\partial U}{\partial S}\right)_{V,N} = T(S,V)$ and $\left(\frac{\partial U}{\partial V}\right)_{S,N} = -p(S,V)$ are independent functions.

(c) In this case $\delta Q = 0$, $\delta W = 0$ and hence $dU = 0$. Consequently, because of the second law, $dS \geq \frac{\delta Q}{T} = 0$, i.e., the entropy can only increase in non-equilibrium and is, therefore, maximal when the system reaches the equilibrium state.

(d) Legendre transform: $F = U - \left(\frac{\partial U}{\partial S}\right)_{V,N} S = U - TS$. In differential form $dF = dU - TdS - SdT = -SdT - pdV$.
In the described situation, $dT = 0$, $dV = 0$ and thus $dU = \delta Q$. Second law dictates $TdS \geq \delta Q = dU$. Since $dT = 0$: $TdS = d(TS)$. Therefore, the second law now reads: $d(U - TS) = dF \leq 0$. Consequently, F can only decrease in this scenario until it reaches its minimum in equilibrium.