Prelim Exam, Day 2, January 4, 2013

Read all of the following information before starting the exam.

Begin each question on a new sheet of paper. Write on only one side of each sheet. Use as many sheets as you need, writing your name and problem number in the top left corner of every sheet. Staple all the sheets for the same question together as you hand them in.

The exam is closed book and closed notes. You may refer only to the formula sheet provided with this copy of the test and a hand book of integral tables which will be at the proctor's table.

Calculators are permitted. You may not use the calculator application of your smart phones.

You may not engage in written or oral communications with anyone during the test period, even if the communication is not germane to physics.

You are reminded that you agreed to abide by UGA's academic honesty policy and procedures, known as *A Culture of Honesty*, when you applied for admission to the University of Georgia.

If you believe there exist circumstances that grievously affect your ability to take this exam, you are obliged to raise your concern with the proctor *before* the start of the exam.

Show all your work for full credit.

Problem 2.1

Electricity and Magnetism

Charge is distributed in a spherically symmetric matter according to the law:

$$\rho(r) = A(1 - \frac{r}{a}), r \leq a$$

and

$$\rho(r) = 0, r > a$$

Calculate the potential energy associated with this charge.

Problem 2.2

Quantum Mechanics (four parts)

The bound state energy of the hydrogenic system is given by $-R(Z/n)^2$, where $R = 13.6$ eV, Z the nuclear charge, $n = 1, 2, ...$ the principal quantum number. Let us apply this system to describe the Helium atom ignoring the interaction between the electrons.

a. What is the ground state energy of ^4He due to the electrons if the hydrogenic system is applied?

b. What would be the energy of the first excited state?

c. Would the energy be higher or lower if the interaction between the electrons were allowed?

d. Is the Pauli principle being violated in this description?

Problem 2.3

Quantum Mechanics: Conservation of linear momentum (four parts)

Consider Quantum mechanics in one dimension x.

The system has a particle of mass m in a potential V(x) for all x.

a. State the linear momentum operator $\hat{P}_x$.

b. State the expectation value $< \hat{P}_x >$ in terms of the normalized wavefunction $\Psi(x)$ and $\hat{P}_x$.

c. Evaluate the time derivative of $< \hat{P}_x >$ to show the dependence on the commutator $[\hat{P}_x, \hat{H}]$ with the Hamiltonian $\hat{H}$.

d. Evaluate the expectation value of the commutator $< [\hat{P}_x, \hat{H}] >$ to show the dependence on V(x).

You must show all the steps in the calculations to receive full credit.

Problem 2.4

Quantum Mechanics (three parts)

1. Consider LS coupling for a multi-electron atom in the 5D term. If $\vec{L}$ is the total orbital angular momentum and $\vec{S}$ the total spin angular momentum of the electrons, the total angular momentum is $\vec{J} = \vec{L} + \vec{S}$.

(a)What are the possible values of J?

(b) If the spin-orbit interaction is given by the Hamiltonian $H = A(\vec{L} \cdot \vec{S})$, determine a relation for the spin-orbit splitting energy in terms of A, J, L, and S.

(c) Taking A to be a positive constant, draw roughly to scale an energy level diagram labeling the ${}^5D_J$ levels with energies respect to the term energy.

Problem 2.5

Electricity and Magnetism (three parts)

Charges $3q$, $-q$, and $-q$ are placed on a straight line at points A, B, and C, respectively, where B is the midpoint of AC.

(a) Draw a rough diagram of the lines of force.

(b) Show that an electric field line that starts from A making an angle with AB that is greater than $\cos^{-1}(-2/3)$ will not reach B or C.

(c) Show that the asymptote of the line of force for which $\Theta = \cos^{-1}(-2/3)$ is at right angles to AC.

Problem 2.6

Electricity and Magnetism (3 parts)

Two non-intersecting spheres, A and B, of radii R_A and R_B, contain electrical charges, Q_A and Q_B, respectively. Each charge is spread out as a uniform charge density in the interior volume of A and B, respectively. In the space outside of both spheres, the electrical charge density is zero. Relative to a cartesian coordinate system, with position vectors denoted by $\vec{r} = (x, y, z)$, the centers of the charged spheres are located at $\vec{r}_A = (0, 0, 0)$ and $\vec{r}_B = (L/2, L/2, 0)$, respectively. Here L denotes the side length of a cube C whose edges are parallel to the three coordinate axes, with four of the cube's eight corners being located at $(0, 0, 0)$, $(L, 0, 0)$ $(0, L, 0)$ and $(0, 0, L)$. Note that $\vec{r}_A$ is one of these corners and that $\vec{r}_B$ is one of the face centers of cube C. You may assume that $L/\sqrt{2} > R_A + R_B$ so that the two spheres do not intersect.

Let S denote the surface of the cube C, *i.e.*, S is the closed surface which consists of all six $L \times L$ square faces of the cube. Also, let $\vec{E}(\vec{r})$ denote the electric field generated by the two charged spheres at position $\vec{r}$.

(a) Make a careful sketch of the cube C and of the two spheres A and B, indicating the location of cube's eight corners relative to the three coordinate axes, and the location of the two spheres, relative to the cube's corners and faces.

(b) Use Gauss's Law to find the total electrical flux through the cube surface S, defined by the surface integral $\Phi = \oint_S \vec{E} \cdot d\vec{a}$ where the infinitesimal surface-normal area vectors $d\vec{a}$ are chosen to point outward at every point on the surface S. Express your result in terms of Q_A, Q_B, R_A, R_B, L, and the permittivity of free space, ϵ_o. Does Φ depend on R_A, R_B, and/or L?

(c) Assume $Q_A = 2\mu\text{C}$, $Q_B = 8\mu\text{C}$, $R_A = 4\text{m}$, $R_B = 1\text{m}$, $L = 7.5\text{m}$, and $\epsilon_o = 8.85 \times 10^{-12}\text{C}^2/(\text{Nm}^2)$. Find the cartesian components of the electric field vector $\vec{E} \equiv (E_x, E_y, E_z)$ at the point $\vec{r} = (0, 0, 3\text{m})$. Hint: Use Gauss's Law to first find the electric field $\vec{E}_A$ generated by sphere A and the electric field $\vec{E}_B$ generated by sphere B, at point $\vec{r}$.

Useful Integrals

$$\int \sin^2\left(\frac{n\pi}{L}x\right) dx = \frac{x}{2} - \frac{L}{4n\pi}\sin\left(\frac{2n\pi}{L}x\right)$$

$$\int x\sin^2\left(\frac{n\pi}{L}x\right) dx = \frac{x^2}{4} - \frac{Lx}{4n\pi}\sin\left(\frac{2n\pi}{L}x\right) - \frac{L^2}{8n^2\pi^2}\cos\left(\frac{2n\pi}{L}x\right)$$

$$\int x^2\sin^2\left(\frac{n\pi}{L}x\right) dx = \frac{x^3}{6} - \frac{Lx^2}{4n\pi}\sin\left(\frac{2n\pi}{L}x\right) - \frac{L^2x}{4n^2\pi^2}\cos\left(\frac{2n\pi}{L}x\right) + \frac{L^3}{8n^3\pi^3}\sin\left(\frac{2n\pi}{L}x\right)$$

$$\int_0^\infty x^m e^{-bx}\, dx = \frac{m!}{b^{m+1}}$$

Gaussian Integrals

$$\int_{-\infty}^{+\infty} e^{-a(z-b)^2}\, dz = \sqrt{\frac{\pi}{a}} \qquad \int_{-\infty}^{+\infty} z e^{-a(z-b)^2}\, dz = b\sqrt{\frac{\pi}{a}}$$

$$\int_{-\infty}^{+\infty} e^{-az^2+bz}\, dz = e^{b^2/4a}\sqrt{\frac{\pi}{a}} \qquad \int_{-\infty}^{+\infty} z^2\, e^{-az^2}\, dz = \frac{1}{2}\sqrt{\frac{\pi}{a^3}}$$

$$\int_{-\infty}^{+\infty} z^{n+2}\, e^{-az^2}\, dz = -\frac{d}{da}\int_{-\infty}^{+\infty} z^n\, e^{-az^2}\, dz$$

Note Sheet for Physics Qualifying Exam

Vector Identities: (a, b, c, d, and F are vector fields, and ψ is a scalar field)

$$\nabla\times\nabla\psi = 0$$
$$\nabla\cdot(\nabla\times\mathbf{a}) = 0$$
$$\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a}$$
$$\nabla\cdot(\psi\mathbf{a}) = \mathbf{a}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{a}$$
$$\nabla\times(\psi\mathbf{a}) = \nabla\psi\times\mathbf{a} + \psi\nabla\times\mathbf{a}$$
$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})$$
$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$$
$$\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b}$$

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$
$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$$
$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

Vector Differential Operators:

Cylindrical coordinates: (ρ, ϕ, z)

$$\nabla\psi = \frac{\partial\psi}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\hat{\phi} + \frac{\partial\psi}{\partial z}\hat{z}$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

$$\nabla\cdot\mathbf{F} = \frac{1}{\rho}\frac{\partial(\rho F_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial F_\phi}{\partial\phi} + \frac{\partial F_z}{\partial z}$$

$$\nabla\times\mathbf{F} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ F_\rho & \rho F_\phi & F_z \end{vmatrix}$$

Spherical coordinates: (r, θ, ϕ)

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\hat{\phi}$$

$$\nabla\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial(r^2 F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, F_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}$$

$$\nabla\times\mathbf{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ F_r & rF_\theta & r\sin\theta\, F_\phi \end{vmatrix}$$

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2(r\psi)}{\partial r^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

Trigonometric Identities:

$$\sin(\alpha\pm\beta) = \sin\alpha\,\cos\beta \pm \cos\alpha\,\sin\beta$$
$$\cos(\alpha\pm\beta) = \cos\alpha\,\cos\beta \mp \sin\alpha\,\sin\beta$$

Physical Data and Conversions:

Quantity	Symbol	Value
Speed of light in vacuum	c	2.9979×10^8 m/s
Gravitational constant	G	6.6743×10^{-11} N·m²/kg²
Coulomb constant ($\epsilon_0 \equiv 1/4\pi k$)	k	8.9876×10^9 N·m²/C²
Planck constant ($\hbar \equiv h/2\pi$)	h	6.6261×10^{-34} J·s
Boltzmann constant	k_B	1.3807×10^{-23} J/K
Avogadro number	N_A	6.0221×10^{23} /mol
Molar gas constant	$R \equiv N_A k_B$	8.3145 J/mol·K
Proton rest mass	m_p	1.6726×10^{-27} kg
Electron rest mass	m_e	9.1094×10^{-31} kg
Electron charge magnitude	e	1.6022×10^{-19} C
Electron-volt	1 eV	1.6022×10^{-19} J
Celsius scale offset	0° C	273.15 K

Solar System Physical Data:

Quantity	Value
Earth mass	5.972×10^{24} kg
Earth radius	6.378×10^6 m
Mean Earth-Moon distance	3.844×10^8 m
Mean Earth-Sun distance	1.496×10^{11} m

Quantity	Value
Moon mass	7.348×10^{22} kg
Moon radius	1.737×10^6 m
Solar mass	1.988×10^{30} kg
Solar radius	6.955×10^8 m

Pauli Spin Matrices:

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Definite Integrals: (for integer $n \geq 0$)

$$\int_0^{\pi/2} \cos^{2n} x\, dx = \int_0^{\pi/2} \sin^{2n} x\, dx = \frac{\pi\,(2n)!}{2^{2n+1}\,(n!)^2}$$

$$\int_0^{\pi/2} \cos^{2n+1} x\, dx = \int_0^{\pi/2} \sin^{2n+1} x\, dx = \frac{2^{2n}(n!)^2}{(2n+1)!}$$

$$\int_0^{\infty} x^{2n} e^{-ax^2}\, dx = \sqrt{\frac{\pi}{a}}\, \frac{(2n)!}{2^{2n+1} a^n\, n!}$$

$$\int_0^{\infty} x^{2n+1} e^{-ax^2}\, dx = \frac{n!}{2a^{n+1}}$$