Prelim Exam, Day 1, January 3, 2013

Read all of the following information before starting the exam.

Begin each question on a new sheet of paper. Write on only one side of each sheet. Use as many sheets as you need, writing your name and problem number in the top left corner of every sheet. Staple all the sheets for the same question together as you hand them in.

The exam is closed book and closed notes. You may refer only to the formula sheet provided with this copy of the test and a hand book of integral tables which will be at the proctor's table.

Calculators are permitted. You may not use the calculator application of your smart phones.

You may not engage in written or oral communications with anyone during the test period, even if the communication is not germane to physics.

You are reminded that you agreed to abide by UGA's academic honesty policy and procedures, known as *A Culture of Honesty*, when you applied for admission to the University of Georgia.

If you believe there exist circumstances that grievously affect your ability to take this exam, you are obliged to raise your concern with the proctor *before* the start of the exam.

Show all your work for full credit.

Problem 1.1

Classical Mechanics (four parts)

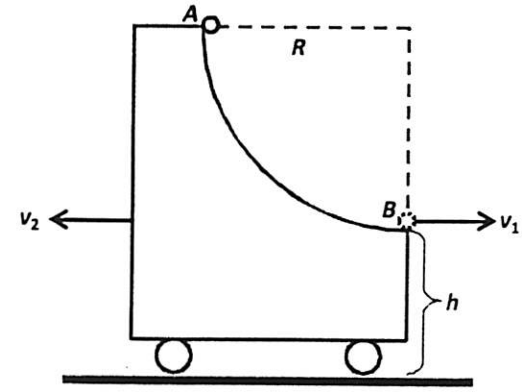
A cart of mass M consists of a frictionless arc AB, which is a quarter-circle with radius R. The height from point B to the floor is h. The cart is free to move along the horizontal floor without friction. A solid ball of mass m is released from rest at point A, as shown in the figure.

(1) What is the velocity v_1 of the solid ball at point B?

(2) From point A to point B, find the horizontal movement Δ of the cart.

(3) From point B to the floor, find the horizontal movement s of the solid ball.

(4) At point B, find the normal supporting force acted on the solid ball.



Problem 1.2

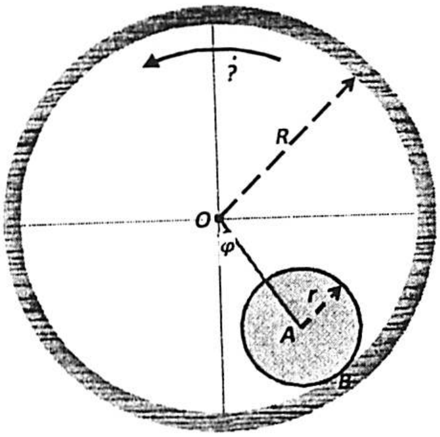
Mechanics (three parts)

A small solid cylinder of radius r and mass m rolls on the inside of a large hollow cylinder of inner radius R and mass M without relative slipping. The hollow cylinder rotates about a fixed horizontal axis with angular velocity $\dot{\Theta}$. The position of the solid cylinder is determined by ϕ , which is the angle of OA respect to the vertical direction, as shown in the figure.

(1) Find angular velocity ω of the solid cylinder (Give your answer in terms of r, R, $\dot{\Theta}$, and the angular velocity $\dot{\phi}$ of OA);

(2) Find the differential equation of motion of the system (Hint: the moment of inertia of the solid cylinder is $I = \frac{1}{2}mr^2$);

(3) Assume $\phi << 1$, find the swing period of the solid cylinder.



Problem 1.3

Modern Physics: Time dilation in the theory of special relativity (2 parts).

a. State the two Einstein's postulates of special relativity:
   (Hints:)
   1. related to laws of physics and inertial reference frames,
   2. on speed of light in vacuum and inertial reference frames.

b. Derive the Lorentz factor for time dilation by considering two identical light clocks in two different reference frames.

The light clock is a tube of length L with mirrors at each end, aligned in the y direction, perpendicular to the relative direction of motion of the two reference frames. A light pulse bounces back and forth between the two mirrors. One tick of the clock is the time interval for one round-trip of the light pulse.

One clock (A) is at rest in a reference frame (A) that is also at rest. The time interval for one tick as observed in the rest frame A for clock A is denoted as Δt_o.

Another identical clock (B) is moving but at rest in the moving reference frame (B) that is moving in the x direction with a speed of v. Observer in the rest frame (A) will measure a different time interval of the B moving clock's tick. This is denoted as Δt.

Draw a sketch of the gedanken experiment.

Invoke the two postulates of part a and derive the relation between the two time intervals. This is an equation for Δt in terms of Δt_o, v and c. It is independent of L. c is the speed of light in vacuum.

Be careful to state your reasoning clearly and draw diagrams to explain your solution. Writing down the equation is not sufficient to receive full credit.

Problem 1.4

Thermodynamics (2 parts)

a. Define the *Helmholtz* and *Gibbs* free energies. Why do we need them both and how do they differ from each other.

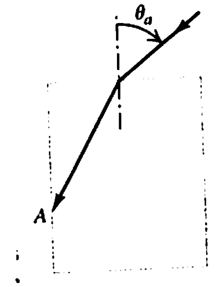
b. Two systems (say water and ice) are in phase equilibrium. What are the condition(s) for the phase equilibrium. Draw the phase diagram in P vs V and P vs T. (P pressure, V volume, T temperature).

Problem 1.5

Optics

A ray of light is incident in air on a block of a transparent solid. For incident angles larger than $\Theta_a = 30^o$, total internal reflection will occur at the vertical face (point A in the figure below).

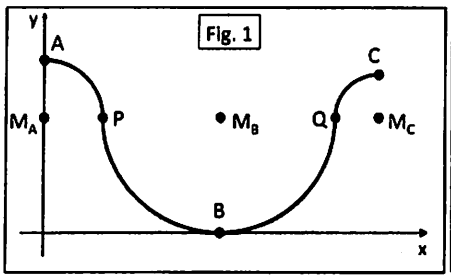
What is the index of refraction of the transparent solid?

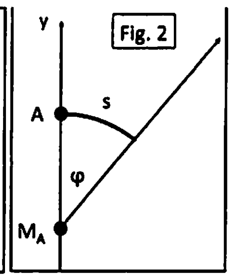


Problem 1.6

Classical Mechanics (3 parts)

A roller coaster track, shown below in Fig. 1, consists of three circular arc segments, AP, PBQ, and QC, where AP and QC are quarter-circles and PBQ is a semi-circle. The corresponding full cirles are centered at points M_A, M_B and M_C, respectively; and their radii are r_A, r_B and r_C, respectively. The arc segments are joined with vertical slopes at points P and Q. Points $A \equiv (x_A, y_A)$ and M_A are located on the vertical y-axis, *i.e.*, $x_A = 0$; and point $B \equiv (x_B, y_B)$ is located on the horizontal x-axis, *i.e.*, $y_B = 0$.





The roller coaster car, including passengers, has a mass m and it is constrained to move only tangentially to the track, *i.e.*, the car is mounted on the track so that it cannot fall off or fly off the track. The size of the car is negligibly small compared to the size of the track. The car starts from A with an initial speed v_A, moving in the $+x$-direction. The car is subject to three forces: (1) the force $\vec{F}_t$ exerted by the track which constrains the car to move tangentially to the track; (2) the downward force of gravity, $\vec{F}_g$, with a gravitational potential energy mgy where g is the gravitational acceleration; and (3) a friction force $\vec{F}_f$ of constant strength $F_f \equiv |\vec{F}_f|$. $\vec{F}_t$ is always normal to the track, *i.e.*, perpendicular to the tangent to the track. $\vec{F}_f$ is always tangential to the track and directed opposite to the car's direction of motion.

(a) Suppose $m = 1000\text{kg}$, $r_A = 10\text{m}$, $r_B = 20\text{m}$, $r_C = 9\text{m}$, $g = 9.81\text{m/s}^2$, $v_A = 4\text{m/s}$, and $F_f = 250\text{N}$. Will the car reach point C? If so, what is the car's speed, v_C, at point C? If not, what is the minimum initial speed, $v_{A,\text{min}}$, which the car must have at A in order to just barely reach point C?

(b) Let s denote the car's travel distance along the track segment AP, *i.e.*, s is the arc length traveled, starting from point A, as shown in Fig. 2. Find the car's speed v along segment AP, expressed as a function of s and as a function of the parameters v_A, r_A, m, g, and F_f. Do not plug in values for these five parameters. Hint: The length s of a circular arc of radius r_A, spanning an angle φ, is $s = r_A\,\varphi$, with φ measured in radians.

(c) How long does it take for the car to travel from A to P? State your result in a general formula, as a definite integral over the travel distance s, with the integrand expressed as a function of s and the parameters v_A, r_A, m, g, and F_f; and with the upper integration limit expressed in terms of r_A. Do not plug in values for the latter five parameters. Do not try to carry out the integration.

Useful Integrals

$$\int \sin^2\left(\frac{n\pi}{L}x\right) dx = \frac{x}{2} - \frac{L}{4n\pi}\sin\left(\frac{2n\pi}{L}x\right)$$

$$\int x\sin^2\left(\frac{n\pi}{L}x\right) dx = \frac{x^2}{4} - \frac{Lx}{4n\pi}\sin\left(\frac{2n\pi}{L}x\right) - \frac{L^2}{8n^2\pi^2}\cos\left(\frac{2n\pi}{L}x\right)$$

$$\int x^2\sin^2\left(\frac{n\pi}{L}x\right) dx = \frac{x^3}{6} - \frac{Lx^2}{4n\pi}\sin\left(\frac{2n\pi}{L}x\right) - \frac{L^2x}{4n^2\pi^2}\cos\left(\frac{2n\pi}{L}x\right) + \frac{L^3}{8n^3\pi^3}\sin\left(\frac{2n\pi}{L}x\right)$$

$$\int_0^\infty x^m e^{-bx}\, dx = \frac{m!}{b^{m+1}}$$

Gaussian Integrals

$$\int_{-\infty}^{+\infty} e^{-a(z-b)^2}\, dz = \sqrt{\frac{\pi}{a}} \qquad \int_{-\infty}^{+\infty} z e^{-a(z-b)^2}\, dz = b\sqrt{\frac{\pi}{a}}$$

$$\int_{-\infty}^{+\infty} e^{-az^2+bz}\, dz = e^{b^2/4a}\sqrt{\frac{\pi}{a}} \qquad \int_{-\infty}^{+\infty} z^2\, e^{-az^2}\, dz = \frac{1}{2}\sqrt{\frac{\pi}{a^3}}$$

$$\int_{-\infty}^{+\infty} z^{n+2}\, e^{-az^2}\, dz = -\frac{d}{da}\int_{-\infty}^{+\infty} z^n\, e^{-az^2}\, dz$$

Note Sheet for Physics Qualifying Exam

Vector Identities: (a, b, c, d, and F are vector fields, and ψ is a scalar field)

$$\nabla\times\nabla\psi = 0$$
$$\nabla\cdot(\nabla\times\mathbf{a}) = 0$$
$$\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a}$$
$$\nabla\cdot(\psi\mathbf{a}) = \mathbf{a}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{a}$$
$$\nabla\times(\psi\mathbf{a}) = \nabla\psi\times\mathbf{a} + \psi\nabla\times\mathbf{a}$$
$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})$$
$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$$
$$\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b}$$

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$
$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$$
$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

Vector Differential Operators:

Cylindrical coordinates: (ρ, ϕ, z)

$$\nabla\psi = \frac{\partial\psi}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\hat{\phi} + \frac{\partial\psi}{\partial z}\hat{z}$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

$$\nabla\cdot\mathbf{F} = \frac{1}{\rho}\frac{\partial(\rho F_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial F_\phi}{\partial\phi} + \frac{\partial F_z}{\partial z}$$

$$\nabla\times\mathbf{F} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ F_\rho & \rho F_\phi & F_z \end{vmatrix}$$

Spherical coordinates: (r, θ, ϕ)

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\hat{\phi}$$

$$\nabla\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial(r^2F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, F_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}$$

$$\nabla\times\mathbf{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ F_r & rF_\theta & r\sin\theta\, F_\phi \end{vmatrix}$$

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2(r\psi)}{\partial r^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

Trigonometric Identities:

$$\sin(\alpha\pm\beta) = \sin\alpha\,\cos\beta \pm \cos\alpha\,\sin\beta$$
$$\cos(\alpha\pm\beta) = \cos\alpha\,\cos\beta \mp \sin\alpha\,\sin\beta$$

Physical Data and Conversions:

Quantity	Symbol	Value
Speed of light in vacuum	c	2.9979×10^8 m/s
Gravitational constant	G	6.6743×10^{-11} N·m²/kg²
Coulomb constant ($\epsilon_0 \equiv 1/4\pi k$)	k	8.9876×10^9 N·m²/C²
Planck constant ($\hbar \equiv h/2\pi$)	h	6.6261×10^{-34} J·s
Boltzmann constant	k_B	1.3807×10^{-23} J/K
Avogadro number	N_A	6.0221×10^{23} /mol
Molar gas constant	$R \equiv N_A k_B$	8.3145 J/mol·K
Proton rest mass	m_p	1.6726×10^{-27} kg
Electron rest mass	m_e	9.1094×10^{-31} kg
Electron charge magnitude	e	1.6022×10^{-19} C
Electron-volt	1 eV	1.6022×10^{-19} J
Celsius scale offset	0° C	273.15 K

Solar System Physical Data:

Quantity	Value
Earth mass	5.972×10^{24} kg
Earth radius	6.378×10^6 m
Mean Earth-Moon distance	3.844×10^8 m
Mean Earth-Sun distance	1.496×10^{11} m

Quantity	Value
Moon mass	7.348×10^{22} kg
Moon radius	1.737×10^6 m
Solar mass	1.988×10^{30} kg
Solar radius	6.955×10^8 m

Pauli Spin Matrices:

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Definite Integrals: (for integer $n \geq 0$)

$$\int_0^{\pi/2} \cos^{2n} x \, dx = \int_0^{\pi/2} \sin^{2n} x \, dx = \frac{\pi\,(2n)!}{2^{2n+1}\,(n!)^2}$$

$$\int_0^{\pi/2} \cos^{2n+1} x \, dx = \int_0^{\pi/2} \sin^{2n+1} x \, dx = \frac{2^{2n}(n!)^2}{(2n+1)!}$$

$$\int_0^{\infty} x^{2n} e^{-ax^2} \, dx = \sqrt{\frac{\pi}{a}}\,\frac{(2n)!}{2^{2n+1} a^n\, n!}$$

$$\int_0^{\infty} x^{2n+1} e^{-ax^2} \, dx = \frac{n!}{2a^{n+1}}$$