PHYS 1312: Potential due to a charged rod
Chapter 16
Nov. 1, 2018

Problem. A rod of length L located on the x-axis with one end at the origin and the other end at $< L, 0, 0 >$ has a total charge Q uniformly distributed. Find the electric potential at $< 0, a, 0 >$.

Solution. Start with the integral equation for the electric potential at r for a continuous charge distribution

$$V(r) = k_e \int \frac{dq}{r}. \tag{1}$$

We need to evaluate i) dq, ii) r, and (iii) the quantity to integrate and its limits. Take the observation location to be on the y-axis, $\vec{r}_{\text{obs}} =< 0, a, 0 >$, while $\vec{r}_{\text{source}} =< x, 0, 0 >$. Then we have for the radial vector

$$\vec{r} = \vec{r}_{\text{obs}} - \vec{r}_{\text{source}} =< -x, a, 0 >, \tag{2}$$

giving $r = \sqrt{x^2 + a^2}$. Now the total charge is related to the linear charge density by $Q = \lambda L$. The charge differential is then

$$dq = \lambda dx. \tag{3}$$

Substitution of all the above in Eq. 1 gives

$$V(a) = k_e \int_0^L \frac{\lambda dx}{\sqrt{x^2 + a^2}}, \tag{4}$$

or

$$V(a) = k_e \frac{Q}{L} \ln(x + \sqrt{x^2 + a^2})\Big|_0^L = k_e \frac{Q}{L}[\ln(L + \sqrt{L^2 + a^2}) - \ln a] \tag{5}$$

or

$$V = k_e \frac{Q}{L} \ln\left(\frac{L + \sqrt{L^2 + a^2}}{a}\right). \tag{6}$$

For $a >> L$, use the series expansion for ln

$$\ln(1 + z) = z - \frac{z^2}{2} + ... \tag{7}$$

giving

$$V \approx k_e \frac{Q}{L} \ln\left(\frac{L + a}{a}\right) = k_e \frac{Q}{L} \ln(1 + L/a) \tag{8}$$

giving the potential for a point charge

$$V \approx k_e \frac{Q}{L} \frac{L}{a} = \frac{k_e Q}{a}. \tag{9}$$

Electric potentials for other charge distributions: (i) ring of radius R with observation on the z-axis,

$$V_{\text{ring}} = \frac{k_e Q}{\sqrt{R^2 + z^2}}; \tag{10}$$

disk of radius R and surface area A with observation on the z-axis,

$$V_{\text{disk}} = \frac{1}{2\epsilon_0}\frac{Q}{A}[\sqrt{R^2+z^2}-z]; \tag{11}$$

and spherical shell of radius R,

$$V_{\text{shell}} = \frac{k_e Q}{r}, r > R, \tag{12}$$

$$V_{\text{shell}} = \frac{k_e Q}{R}, r < R. \tag{13}$$