PHYS 1312: Example Problem Solution
Chapter 16
Nov. 1, 2018

Problem P86. A rod uniformly charged with charge $-q$ is bent into a semicircular arc of radius b in the $x-y$ plane, as shown in Figure 16.97 of the textbook (arc is from $\theta = \pi/2$ to $3\pi/2$ centered on the origin). What is the potential relative to infinity ($V_A = 0$) for a location B on the z-axis through the origin?

Solution. Start with the integral equation for the electric potential at r for a continuous charge distribution

$$V(r) = k_e \int \frac{dq}{r}. \tag{1}$$

We need to evaluate i) dq, ii) r, and (iii) the quantity to integrate and its limits. Take the observation location to be on the z-axis, $\vec{r}_{\rm obs} =< 0, 0, z >$. From the diagram $\vec{r}_{\rm source} =< b\cos\theta, b\sin\theta, 0 >$ with θ measured as usual from the positive x-axis in the $x-y$ plane. Then we have for the radial vector

$$\vec{r} = \vec{r}_{\rm obs} - \vec{r}_{\rm source} =< -b\cos\theta, -b\sin\theta, z > \tag{2}$$

giving $r = \sqrt{b^2 + z^2}$. Now the total charge is related to the linear charge density by $q = \lambda L$ where L is the arc length given by $b\theta$. Or

$$q = \lambda b \theta_{\rm max} \tag{3}$$

where $\theta_{\rm max}$ is the angular range for the charge distribution. The charge differential is then

$$dq = \lambda b d\theta. \tag{4}$$

Substitution of all the above in Eq. 1 gives

$$V(z) = k_e \int_{\pi/2}^{3\pi/2} \frac{\lambda b d\theta}{\sqrt{b^2 + z^2}}, \tag{5}$$

or

$$V(z) = \frac{k_e b\lambda}{\sqrt{b^2 + z^2}} \int_{\pi/2}^{3\pi/2} d\theta = \frac{k_e b\lambda}{\sqrt{b^2 + z^2}} [\theta] \Big|_{\pi/2}^{3\pi/2} \tag{6}$$

or

$$V(z) = \frac{k_e b(q/b/\pi)\pi}{\sqrt{b^2 + z^2}} = \frac{k_e q}{\sqrt{b^2 + z^2}}. \tag{7}$$

For $z = 0$, we get

$$V_B = \frac{k_e q}{b}, \tag{8}$$

while for $z \to \infty$, $V_A = 0$. From Eq. 7, we can obtain the electric field by taking its gradient. Only the z-term is non-zero

$$E_z = -\frac{\partial V}{\partial z} = -k_e q \frac{\partial}{\partial z} \frac{1}{\sqrt{b^2 + z^2}}, \tag{9}$$

or

$$E_z = \frac{k_e q z}{(b^2 + z^2)^{3/2}}, \tag{10}$$

in agreement with the result from the textbook on page 598.