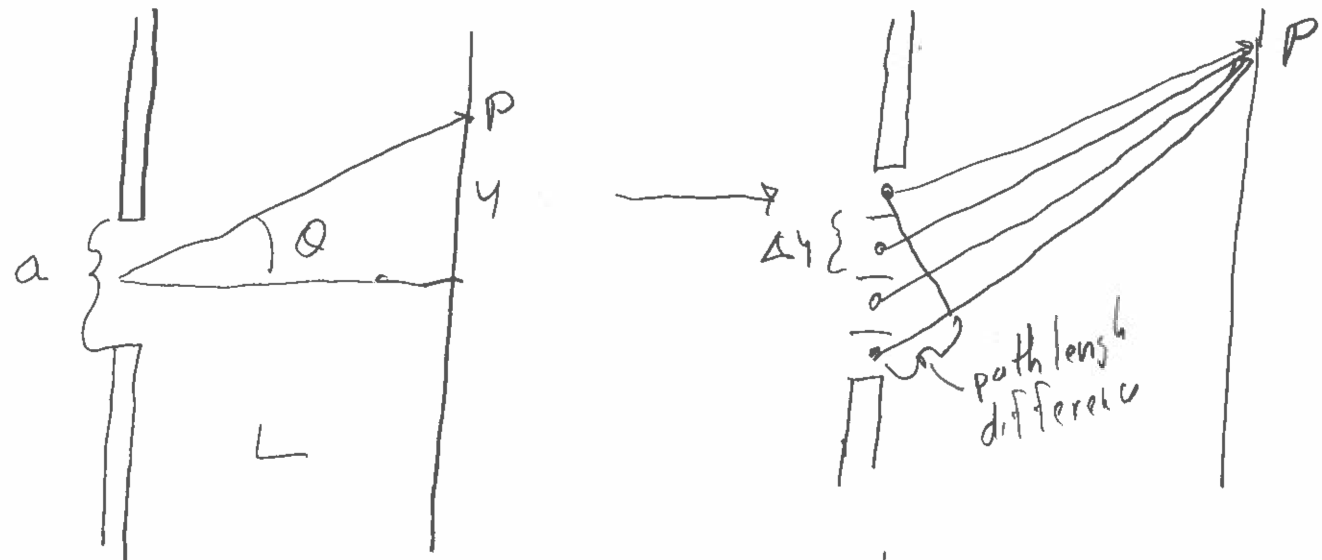
Interference Pattern for the Single Slit

10-6-22



- Apply Hygens Principle in a similar way as done for the double slit, but assume N number of spherical wave emitters
- Divide slit of width a into N sectors of width Δy $\longrightarrow$ $a = N\Delta y$
- All waves emit in phase and have the same electric field amplitude ΔE so that $E = \sum_{i=1}^{N} \Delta E$
  - However each wave travels a different pathlength resulting in a different phase when they arrive at point P
- following the double slit

$$\phi = \frac{2\pi}{\lambda}\delta = \frac{2\pi}{\lambda} d\sin\theta$$

$$\Delta\beta = \frac{2\pi}{\lambda}\Delta y \sin\theta$$

for the phase difference from 1 wave to the next

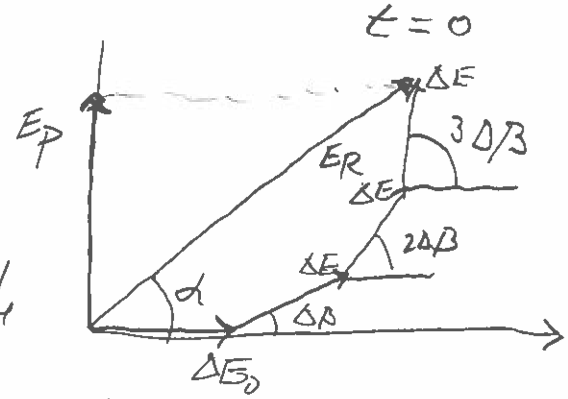
- where $\Delta\beta$ is the phase difference between successive waves originating from each Δy starting at the top with $\Delta\beta = 0$
- Consider $N = 4$ as in the plot, we get four waves interferring at point P with the electric field from each given by

$$E_1 = \Delta E_0 \sin(\omega t), \quad E_2 = \Delta E_0 \sin(\omega t + \Delta\beta)$$

$$E_3 = \Delta E_0 \sin(\omega t + 2\Delta\beta), \quad E_4 = \Delta E_0 \sin(\omega t + 3\Delta\beta)$$

- Draw these electric fields on a phasor diagram since the y-component is the total electric field at point P. Take $t = 0$ for simplicity

$$E_P = E_{1y} + E_{2y} + E_{3y} + E_{4y}$$



- E_R is the resultant magnitude
- $E_P = E_R \sin\alpha$ and the intensity is proportional to E_P^2, $I = (\text{constant}) E_P^2$
- Also $\beta = N\Delta\beta = N\frac{2\pi}{\lambda}\Delta y \sin\theta = \boxed{\frac{2\pi}{\lambda} a \sin\theta = \beta}$
- consider the case for large N and $\beta = 2\pi$

$E_R = 0$, so $E_P = 0$

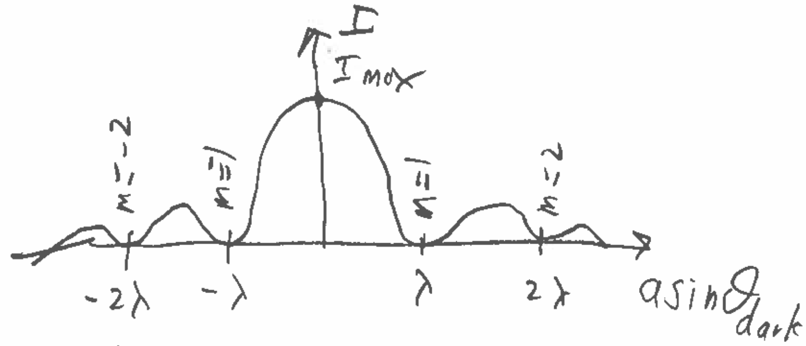
and $I = 0$. This is a minimum

- Minima therefore occur for 10-10-22

$$\beta = \frac{2\pi a \sin\theta}{\lambda} = m\,2\pi\,,\quad m = \pm 1, \pm 2, \pm 3, \ldots$$

or $\boxed{a \sin\theta_{dark} = m\lambda}$

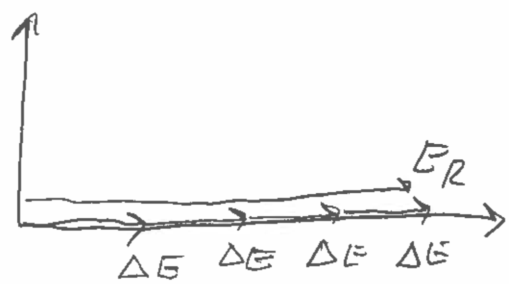
same equation for double slit bright fringes



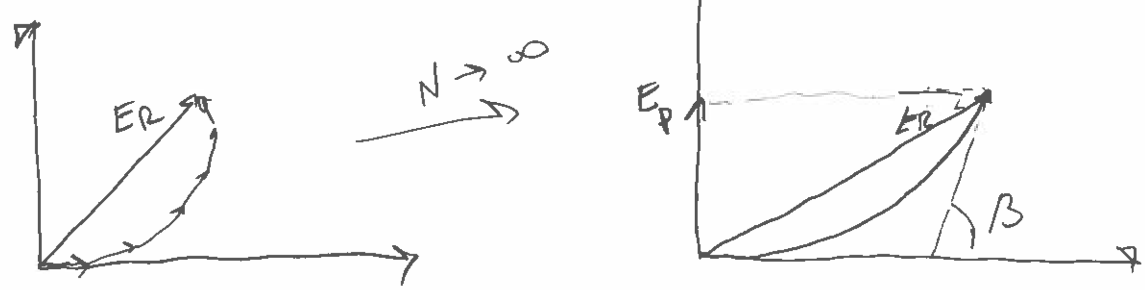
- Note that $m \neq 0$ as that corresponds to the maximum.
- What about the maximum? It occurs for $\theta = 0$
- so all phase differences are zero. For the $N=4$ case, here is the phasor diagram

$E_R = 4\Delta E$ (or in general $N\Delta E$)

- But $E_P = 0$. So, is $I = 0$?



- Return to earlier phasor diagram but consider small θ and let $N \to \infty$
- In this limit, we get



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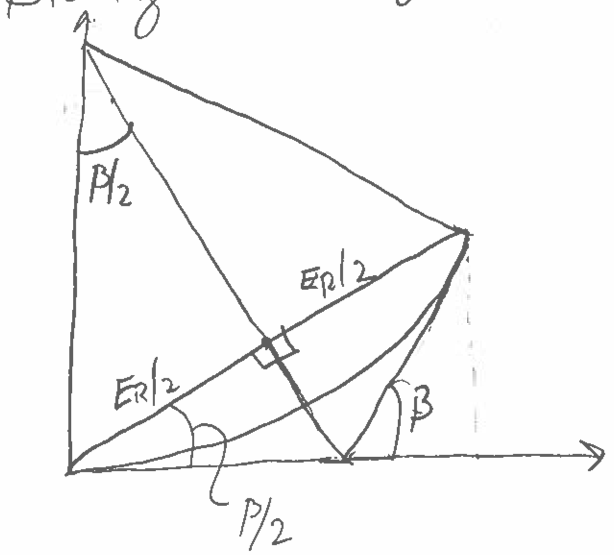
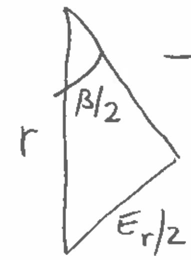
- Blowing that figure up





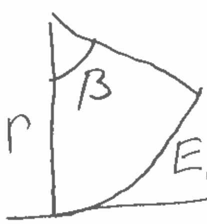
$$\rightarrow \sin\frac{\beta}{2} = \frac{E_r/2}{r}$$

$$E_r = 2r\sin\frac{\beta}{2} \neq f(t)$$

$$180 = 2(90-\alpha) + \beta$$
$$180 = 180 - 2\alpha + \beta$$
$$\alpha = \beta/2$$

$$E_p = E_R \sin\frac{\beta}{2} = 2r\sin\frac{\beta}{2}\sin\frac{\beta}{2}$$

but $E_p = f(t) \Rightarrow E_p = 2r\sin\frac{\beta}{2}\sin(\omega t + \beta/2)$



$$E_0 = \Sigma N \Delta E_0 \equiv \text{arclength} = r\beta \Rightarrow r = \frac{E_0}{\beta}$$

$$\Rightarrow \boxed{E_p = 2\frac{E_0}{\beta}\sin(\beta/2)\sin(\omega t + \beta/2)}$$

$$I = (\text{constant})\, 4\left(\frac{E_0}{\beta}\right)^2 \sin^2(\beta/2)\sin^2(\omega t + \beta/2)$$

$$= I_{max}\left[\frac{\sin(\beta/2)}{\beta/2}\right]^2 \sin^2(\omega t + \beta/2) \rightarrow 1/2$$

So average intensity is

$$I_{avg} = \langle I \rangle = I_{max}\left[\frac{\sin(\beta/2)}{\beta/2}\right]^2$$

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or $\boxed{I_{avg} = I_{max}\left[\frac{\sin(\pi a \sin\theta/\lambda)}{\pi a \sin\theta/\lambda}\right]^2}$

- For $\theta = 0$, $I_{avg} = I_{max}\left[\frac{0}{0}\right]^2$
- So, apply L'Hôpital's rule

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$

here we get $\lim_{\beta/2\to 0}\frac{\sin(\beta/2)}{\beta/2} = \lim_{\beta/2\to 0}\frac{\frac{1}{2}\cos(\beta/2)}{1/2}$

$$= \lim_{\beta/2\to 0}\cos(\beta/2) \to \boxed{1}, \text{ so } I_{avg} \xrightarrow[\beta/2\to 0]{} I_{max}$$

- what about the other maxima? Take derivative of I_{avg} w.r.t β and set to zero

$$\frac{dI}{d\beta} = I_{max}\frac{d}{d\beta}\left[\frac{\sin(\beta/2)}{\beta/2}\right]^2 = 2I_{max}\left\{\frac{\sin(\beta/2)}{\beta/2}\right\}\frac{d}{d\beta}\left[\frac{\sin(\beta/2)}{\beta/2}\right] = 0$$

- This reduces to $(\sin\beta/2)\underbrace{\left[\beta\cos(\beta/2) - 2\sin(\beta/2)\right]} = 0$

roots are $\sin(\beta/2) = 0$, for $\frac{\beta}{2} = m\pi$
or $\boxed{\beta = m2\pi}$ minima as before

roots for maxima, but does not give analytical result, i.e. $\beta \neq (m+\frac{1}{2})2\pi$
It is a transcendental Equation
must solve numerically

maxima

	β	I
1	$\pm 2.860\pi$	$0.0472\, I_{max}$
2	$\pm 4.918\pi$	$0.0165\, I_{max}$
3	$\pm \sim 7\pi$	$0.0083\, I_{max}$

See figure on powerpoint

note no maxima for $\beta \sim 2\pi$