Properties for mass distributions of extended objects 3-23-2004
(Rigid bodies)

- A rigid body or any extended object can be thought of as a collection of a large number of particles

- If the particle separations are very small (Δx) and the masses are very small (Δm), then we can extend the definition of the center of mass

$M = \sum^N \Delta m_i$

$$x_{cm} \approx \frac{\sum_i^N x_i \Delta m_i}{M} \quad \text{for } N \text{ particles of mass } \Delta m_i$$

$$\text{or} \quad x_{cm} = \lim_{\Delta m_i \to 0} \frac{\sum_i^N x_i \Delta m_i}{M} = \frac{1}{M}\int x\,dm \quad \Bigg| \quad M = \sum_{i=1}^N m_i$$

and similar equations for y_{cm} and z_{cm}

which can be combined into $\vec{r}_{cm} = \frac{1}{M}\int \vec{r}\,dm$

Example

Find the center of mass of ruler of mass M and length L

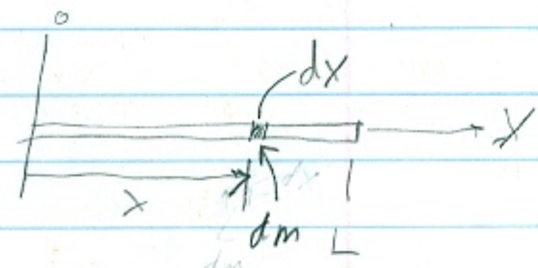
Solution

consider the ruler as 1-D dimensional of length only

3-23-2004

- i.e, ignore width and thickness
- assume constant density
  - but since only one-dimension, we define linear density, $\lambda = \frac{M}{L}$ $\left[\frac{kg}{m}\right]$ (mass per unit length)

or $M = \lambda L \Rightarrow$
$dm = \lambda dx$



$$X_{cm} = \frac{1}{M}\int x\, dm$$

$$= \frac{1}{M}\int_0^L x\, \lambda dx \qquad \lambda = \text{constant}$$

$$= \frac{\lambda}{M}\int_0^L x\,dx = \frac{M}{L}\frac{1}{M}\left[\frac{x^2}{2}\right]_0^L = \frac{1}{L}\left[\frac{L^2}{2}\right] = \underline{\underline{\frac{L}{2}}}$$

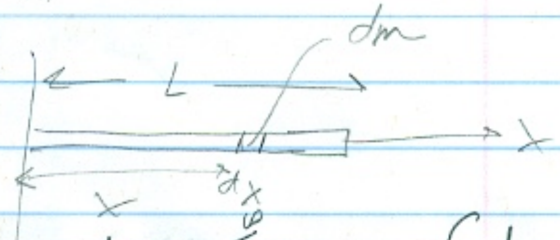
- X_{cm} would not be at the center if $\lambda \neq \text{constant} = f(x)$

Example Problem - Chap. 12 - Rod of length L

$\Rightarrow \lambda(x) = 50.0\,g/m + 20.0\,x\,\frac{g}{m^2}$ $\quad L = 30.0\,cm$

a) $M = ?$

b) $cm = ?$



$dm = \lambda\,dx \qquad M = \Sigma \Delta m = \lim_{\Delta m \to 0} \sum_{i=1} \Delta m_i = \int dm$

$M = \int dm = \int_0^L \lambda\,dx = \int_0^L (50.0 + 20.0x)\,dx$

$= [50x + 10x^2]\Big|_0^L = 50.0L + 10.0L^2 = \underline{15.9\,g}$

$x_{cm} = \frac{1}{M}\int x\,dm = \frac{1}{M}\int_0^L x\,\lambda\,dx$

$= \frac{1}{M}\int_0^L (50x + 20x^2)\,dx$

$= \frac{1}{M}\left[25x^2 + \frac{20x^3}{3}\right]_0^L$

$= \frac{1}{15.9}\left(25(.3)^2 + \frac{20(.3)^3}{3}\right) = 0.153\,m = \boxed{15.3\,cm}$

Calculations of Moments of Inertia 3-23-2004

- while we have Tables to give us the moment of inertia for standard shapes, it is useful to be able to compute them for arbitrary mass distributions

- Following the same procedure as for the center of mass, we replace m_i by Δm_i

$$\lim_{\Delta m_i \to 0} \sum_i r_i^2 \Delta m_i = \boxed{\int r^2 dm = I}$$

moment of inertia integral

- If we have a 3D object, then we can rewrite this in terms of the volume density $\rho = \frac{M}{V}$

$$M = \rho V \rightarrow dm = \rho dV$$

$$I = \int r^2 \rho dV$$

Do uniform rod

Aside

- Consider the following trends

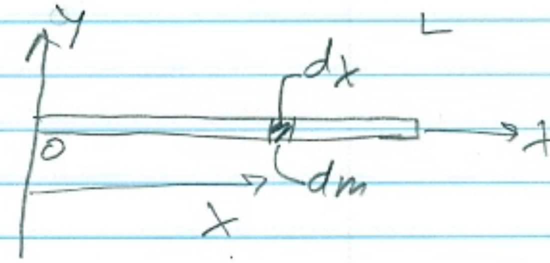
	Point particle(s)	Rigid Body	
Mass	m_i	$M = \int dm$	0-th moment of inertia
CM	$r_{cm} = \sum m_i r_i / M$	$r_{cm} = \frac{1}{M}\int r dm$	1st moment of inertia
I	$\sum m_i r_i^2$	$I = \int r^2 dm$	2nd moment of inertia
general	$\sum m_i r_i^n$ n-th	$\int r^n dm$	

Example - Chapter 12 4-5-2021

Non-uniform density rod

- Determine the moment of inertia for an axis at one end ($L = 30\,cm$)

$\lambda = 50 \frac{g}{m} + 20x \frac{g}{m^2}$



General relation is

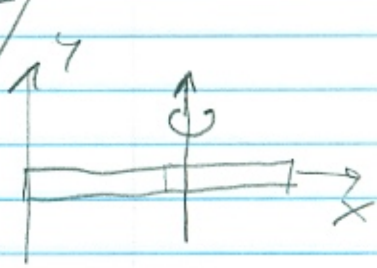
$I = \int r^2 dm$, but 1D problem $\underbrace{I_y}_{\text{rotation axis}} = \int_0^L x^2 dm$ $\qquad dm = \lambda dx$

$$I_y = \int_0^L x^2 \lambda dx = \int_0^L x^2 (50 + 20x) dx$$

$$= 50 \int_0^L x^2 dx + 20 \int_0^L x^3 dx$$

$$= 50 \left. \frac{x^3}{3} \right|_0^L + 20 \left. \frac{x^4}{4} \right|_0^L = \frac{50L^3}{3} + 5L^4$$

$$= \frac{50}{3}(0.3)^3 + 5(0.3)^4 = \boxed{0.4905\ g\,m^2}$$

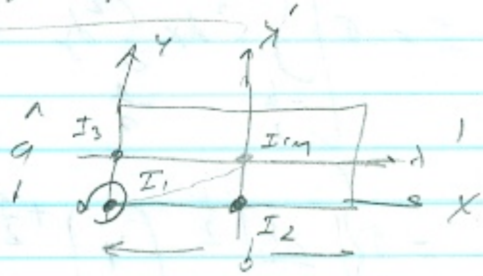


- Moment of inertia about cm

$I = I_{cm} + MD^2$, $\quad D = x_{cm} = 15.3\,cm$

$I_{cm} = I - MD^2 = 0.4905 - (15.9\,g)(.153)^2$
$= \boxed{0.118\ g\,m^2}$

Moment of inertia for rectangular plate



$$I = \int r^2 dm$$

$$= \int (x^2+y^2)\, dm$$

$$\sigma = \frac{M}{A} \equiv \text{area density} \left(\frac{kg}{m^2}\right) \quad = \frac{M}{ab}$$

$$M = \sigma A$$

$$dm = \sigma dA = \sigma dx dy$$

$$I = \iint (x^2+y^2)\, \sigma\, dx dy$$

$$= \sigma\left[\iint x^2 dx dy + \iint y^2 dx dy\right]$$

$$= \sigma\left[\int_0^b x^2 dx \int_0^a dy + \int_0^a y^2 dy \int_0^b dx\right]$$

$$= \sigma\left\{\left(\frac{x^3}{3}\right)\Big|_0^b (x)\Big|_0^a + \left(\frac{y^3}{3}\right)\Big|_0^a x\Big|_0^b\right]$$

$$= \sigma\left[\frac{ab^3}{3} + \frac{a^3b}{3}\right] = \frac{M}{ab}\left[\frac{ab^3}{3} + \frac{a^3b}{3}\right]$$

$$= M\left[\frac{b^2}{3} + \frac{a^2}{3}\right] = \boxed{\frac{M}{3}(a^2+b^2) = I}$$

$$I_{cm} = I - MD^2 \qquad D^2 = (a^2+b^2)/4$$

$$= \frac{1}{3}M(a^2+b^2) - \frac{M(a^2+b^2)}{4} = M(a^2+b^2)\left[\frac{4}{12} - \frac{3}{12}\right] = \boxed{\frac{1}{12}M(a^2+b^2)}$$