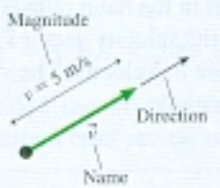
3 Vectors and Coordinate Systems

Wind has both a speed and a direction, hence the motion of the wind is described by a vector.

IN THIS CHAPTER, you will learn how vectors are represented and used.

What is a vector?

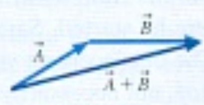
A vector is a quantity with both a size—its magnitude—and a direction. Vectors you'll meet in the next few chapters include position, displacement, velocity, acceleration, force, and momentum.



« LOOKING BACK Tactics Boxes 1.1 and 1.2 Vector addition and subtraction

How are vectors added and subtracted?

Vectors are added "tip to tail." The order of addition does not matter. To subtract vectors, turn the subtraction into addition by writing $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$. The vector $-\vec{B}$ is the same length as $\vec{B}$ but points in the opposite direction.



What are unit vectors?

Unit vectors define what we *mean* by the $+x$- and $+y$-directions in space.

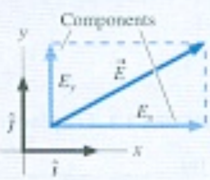
- A unit vector has magnitude 1.
- A unit vector has no units.

Unit vectors simply point.

What are components?

Components of vectors are the pieces of vectors parallel to the coordinate axes—in the directions of the unit vectors. We write

$$\vec{E} = E_x\hat{\imath} + E_y\hat{\jmath}$$



Components simplify vector math.

How are components used?

Components let us do vector math with algebra, which is easier and more precise than adding and subtracting vectors using geometry and trigonometry. Multiplying a vector by a number simply multiplies all of the vector's components by that number.

$$\vec{C} = 2\vec{A} + 3\vec{B}$$

means

$$\begin{cases} C_x = 2A_x + 3B_x \\ C_y = 2A_y + 3B_y \end{cases}$$

How will I use vectors?

Vectors appear everywhere in physics and engineering—from velocities to electric fields and from forces to fluid flows. The tools and techniques you learn in this chapter will be used throughout your studies and your professional career.

3.1 Scalars and Vectors

A quantity that is fully described by a single number (with units) is called a **scalar.** Mass, temperature, volume and energy are all scalars. We will often use an algebraic symbol to represent a scalar quantity. Thus m will represent mass, T temperature, V volume, E energy, and so on.

Our universe has three dimensions, so some quantities also need a direction for a full description. If you ask someone for directions to the post office, the reply "Go three blocks" will not be very helpful. A full description might be, "Go three blocks south." A quantity having both a size and a direction is called a **vector.**

The mathematical term for the length, or size, of a vector is **magnitude,** so we can also say that **a vector is a quantity having a magnitude and a direction.**

FIGURE 3.1 The velocity vector $\vec{v}$ has both a magnitude and a direction.

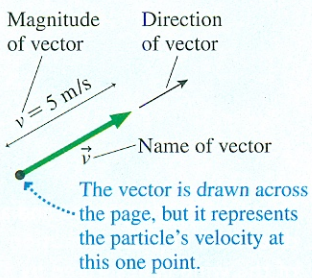


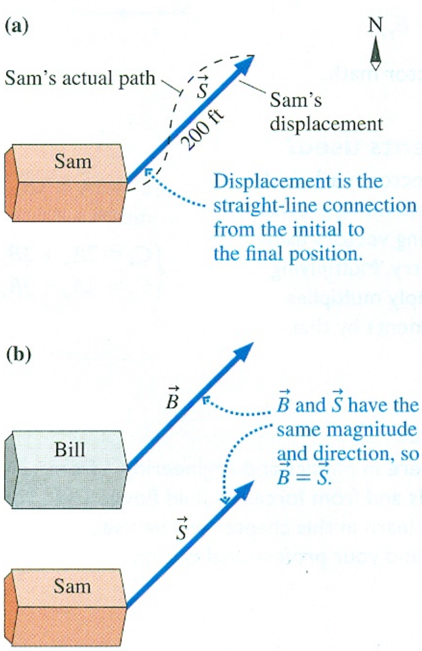
FIGURE 3.1 shows that the *geometric representation* of a vector is an arrow, with the tail of the arrow (not its tip!) placed at the point where the measurement is made. An arrow makes a natural representation of a vector because it inherently has both a length and a direction. As you've already seen, we label vectors by drawing a small arrow over the letter that represents the vector: $\vec{r}$ for position, $\vec{v}$ for velocity, $\vec{a}$ for acceleration.

NOTE Although the vector arrow is drawn across the page, from its tail to its tip, this does *not* indicate that the vector "stretches" across this distance. Instead, the vector arrow tells us the value of the vector quantity only at the one point where the tail of the vector is placed.

The magnitude of a vector can be written using absolute value signs or, more frequently, as the letter without the arrow. For example, the magnitude of the velocity vector in Figure 3.1 is $v = |\vec{v}| = 5$ m/s. This is the object's *speed.* The magnitude of the acceleration vector $\vec{a}$ is written a. **The magnitude of a vector is a scalar.** Note that magnitude of a vector cannot be a negative number; it must be positive or zero, with appropriate units.

It is important to get in the habit of using the arrow symbol for vectors. If you omit the vector arrow from the velocity vector $\vec{v}$ and write only v, then you're referring only to the object's speed, not its velocity. The symbols $\vec{r}$ and r, or $\vec{v}$ and v, do *not* represent the same thing.

FIGURE 3.2 Displacement vectors.



3.2 Using Vectors

Suppose Sam starts from his front door, walks across the street, and ends up 200 ft to the northeast of where he started. Sam's displacement, which we will label $\vec{S}$, is shown in **FIGURE 3.2a**. The displacement vector is a *straight-line connection* from his initial to his final position, not necessarily his actual path.

To describe a vector we must specify both its magnitude and its direction. We can write Sam's displacement as $\vec{S} =$ (200 ft, northeast). The magnitude of Sam's displacement is $S = |\vec{S}| = 200$ ft, the distance between his initial and final points.

Sam's next-door neighbor Bill also walks 200 ft to the northeast, starting from his own front door. Bill's displacement $\vec{B} =$ (200 ft, northeast) has the same magnitude and direction as Sam's displacement $\vec{S}$. Because vectors are defined by their magnitude and direction, **two vectors are equal if they have the same magnitude and direction.** Thus the two displacements in **FIGURE 3.2b** are equal to each other, and we can write $\vec{B} = \vec{S}$.

NOTE A vector is unchanged if you move it to a different point on the page as long as you don't change its length or the direction it points.

Vector Addition

If you earn \$50 on Saturday and \$60 on Sunday, your *net* income for the weekend is the sum of \$50 and \$60. With numbers, the word *net* implies addition. The same is true with vectors. For example, FIGURE 3.3 shows the displacement of a hiker who first hikes 4 miles to the east, then 3 miles to the north. The first leg of the hike is described by the displacement $\vec{A} = (4\text{ mi, east})$. The second leg of the hike has displacement $\vec{B} = (3\text{ mi, north})$. Vector $\vec{C}$ is the *net displacement* because it describes the net result of the hiker's first having displacement $\vec{A}$, then displacement $\vec{B}$.

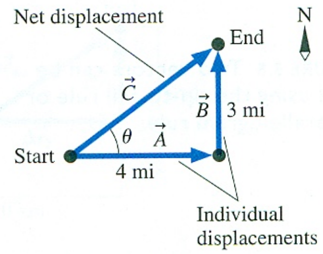


FIGURE 3.3 The net displacement $\vec{C}$ resulting from two displacements $\vec{A}$ and $\vec{B}$.

The net displacement $\vec{C}$ is an initial displacement $\vec{A}$ *plus* a second displacement $\vec{B}$, or

$$\vec{C} = \vec{A} + \vec{B} \qquad (3.1)$$

The sum of two vectors is called the **resultant vector.** It's not hard to show that vector addition is commutative: $\vec{A} + \vec{B} = \vec{B} + \vec{A}$. That is, you can add vectors in any order you wish.

« Tactics Box 1.1 on page 6 showed the three-step procedure for adding two vectors, and it's highly recommended that you turn back for a quick review. This tip-to-tail method for adding vectors, which is used to find $\vec{C} = \vec{A} + \vec{B}$ in Figure 3.3, is called **graphical addition.** Any two vectors of the same type—two velocity vectors or two force vectors—can be added in exactly the same way.

The graphical method for adding vectors is straightforward, but we need to do a little geometry to come up with a complete description of the resultant vector $\vec{C}$. Vector $\vec{C}$ of Figure 3.3 is defined by its magnitude C and by its direction. Because the three vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ form a right triangle, the magnitude, or length, of $\vec{C}$ is given by the Pythagorean theorem:

$$C = \sqrt{A^2 + B^2} = \sqrt{(4\text{ mi})^2 + (3\text{ mi})^2} = 5\text{ mi} \qquad (3.2)$$

Notice that Equation 3.2 uses the magnitudes A and B of the vectors $\vec{A}$ and $\vec{B}$. The angle θ, which is used in Figure 3.3 to describe the direction of $\vec{C}$, is easily found for a right triangle:

$$\theta = \tan^{-1}\left(\frac{B}{A}\right) = \tan^{-1}\left(\frac{3\text{ mi}}{4\text{ mi}}\right) = 37° \qquad (3.3)$$

Altogether, the hiker's net displacement is $\vec{C} = \vec{A} + \vec{B} = (5\text{ mi, } 37°\text{ north of east})$.

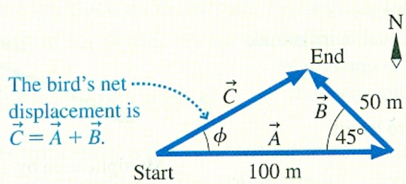
NOTE Vector mathematics makes extensive use of geometry and trigonometry. Appendix A, at the end of this book, contains a brief review of these topics.

EXAMPLE 3.1 Using graphical addition to find a displacement

A bird flies 100 m due east from a tree, then 50 m northwest (that is, 45° north of west). What is the bird's net displacement?

VISUALIZE FIGURE 3.4 shows the two individual displacements, which we've called $\vec{A}$ and $\vec{B}$. The net displacement is the vector sum $\vec{C} = \vec{A} + \vec{B}$, which is found graphically.

FIGURE 3.4 The bird's net displacement is $\vec{C} = \vec{A} + \vec{B}$.



SOLVE The two displacements are $\vec{A} = (100\text{ m, east})$ and $\vec{B} = (50\text{ m, northwest})$. The net displacement $\vec{C} = \vec{A} + \vec{B}$ is found by drawing a vector from the initial to the final position. But describing $\vec{C}$ is a bit trickier than the example of the hiker because $\vec{A}$ and $\vec{B}$ are not at right angles. First, we can find the magnitude of $\vec{C}$ by using the law of cosines from trigonometry:

$$\begin{aligned} C^2 &= A^2 + B^2 - 2AB\cos 45° \\ &= (100\text{ m})^2 + (50\text{ m})^2 - 2(100\text{ m})(50\text{ m})\cos 45° \\ &= 5430\text{ m}^2 \end{aligned}$$

Thus $C = \sqrt{5430\text{ m}^2} = 74\text{ m}$. Then a second use of the law of cosines can determine angle ϕ (the Greek letter phi):

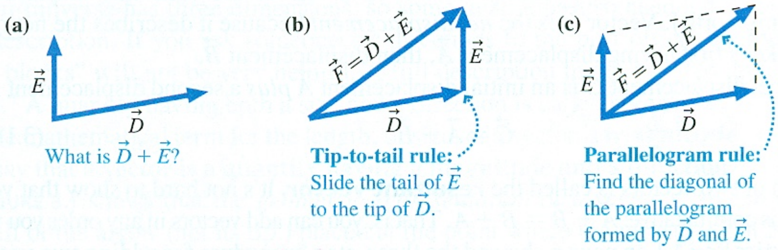
$$B^2 = A^2 + C^2 - 2AC\cos\phi$$

$$\phi = \cos^{-1}\left[\frac{A^2 + C^2 - B^2}{2AC}\right] = 29°$$

The bird's net displacement is

$$\vec{C} = (74\text{ m, } 29°\text{ north of east})$$

It is often convenient to draw two vectors with their tails together, as shown in **FIGURE 3.5a**. To evaluate $\vec{D}+\vec{E}$, you could move vector $\vec{E}$ over to where its tail is on the tip of $\vec{D}$, then use the tip-to-tail rule of graphical addition. That gives vector $\vec{F}=\vec{D}+\vec{E}$ in **FIGURE 3.5b**. Alternatively, **FIGURE 3.5c** shows that the vector sum $\vec{D}+\vec{E}$ can be found as the diagonal of the parallelogram defined by $\vec{D}$ and $\vec{E}$. This method for vector addition is called the *parallelogram rule* of vector addition.



▶ **FIGURE 3.5** Two vectors can be added using the tip-to-tail rule or the parallelogram rule.

Vector addition is easily extended to more than two vectors. **FIGURE 3.6** shows the path of a hiker moving from initial position 0 to position 1, then position 2, then position 3, and finally arriving at position 4. These four segments are described by displacement vectors $\vec{D}_1$, $\vec{D}_2$, $\vec{D}_3$, and $\vec{D}_4$. The hiker's *net* displacement, an arrow from position 0 to position 4, is the vector $\vec{D}_{net}$. In this case,

$$\vec{D}_{net}=\vec{D}_1+\vec{D}_2+\vec{D}_3+\vec{D}_4 \tag{3.4}$$

The vector sum is found by using the tip-to-tail method three times in succession.

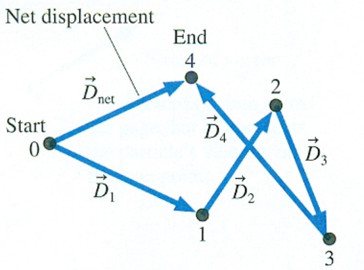
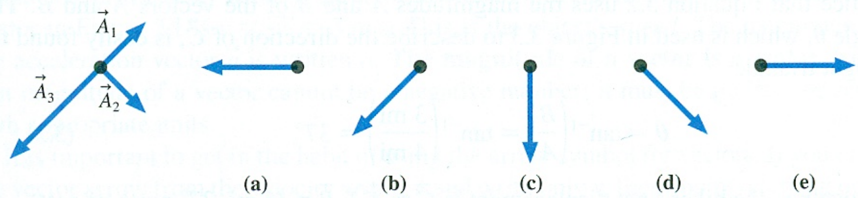


FIGURE 3.6 The net displacement after four individual displacements.

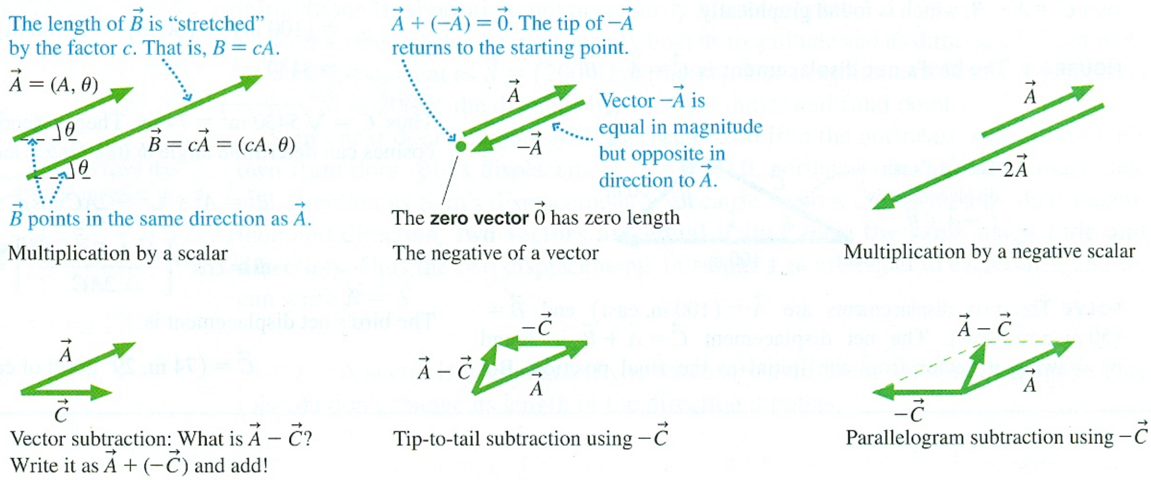
STOP TO THINK 3.1 Which figure shows $\vec{A}_1+\vec{A}_2+\vec{A}_3$?



More Vector Mathematics

In addition to adding vectors, we will need to subtract vectors (« Tactics Box 1.2 on page 7), multiply vectors by scalars, and understand how to interpret the negative of a vector. These operations are illustrated in **FIGURE 3.7**.

FIGURE 3.7 Working with vectors.



EXAMPLE 3.2 **Velocity and displacement**

Carolyn drives her car north at 30 km/h for 1 hour, east at 60 km/h for 2 hours, then north at 50 km/h for 1 hour. What is Carolyn's net displacement?

SOLVE Chapter 1 defined average velocity as

$$\vec{v} = \frac{\Delta\vec{r}}{\Delta t}$$

so the displacement $\Delta\vec{r}$ during the time interval Δt is $\Delta\vec{r} = (\Delta t)\vec{v}$. This is multiplication of the vector $\vec{v}$ by the scalar Δt. Carolyn's velocity during the first hour is $\vec{v}_1 = (30 \text{ km/h, north})$, so her displacement during this interval is

$$\Delta\vec{r}_1 = (1 \text{ hour})(30 \text{ km/h, north}) = (30 \text{ km, north})$$

Similarly,

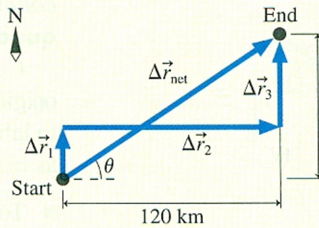
$$\Delta\vec{r}_2 = (2 \text{ hours})(60 \text{ km/h, east}) = (120 \text{ km, east})$$

$$\Delta\vec{r}_3 = (1 \text{ hour})(50 \text{ km/h, north}) = (50 \text{ km, north})$$

In this case, multiplication by a scalar changes not only the length of the vector but also its units, from km/h to km. The direction, however, is unchanged. Carolyn's net displacement is

$$\Delta\vec{r}_{net} = \Delta\vec{r}_1 + \Delta\vec{r}_2 + \Delta\vec{r}_3$$

FIGURE 3.8 The net displacement is the vector sum $\Delta\vec{r}_{net} = \Delta\vec{r}_1 + \Delta\vec{r}_2 + \Delta\vec{r}_3$.



This addition of the three vectors is shown in **FIGURE 3.8**, using the tip-to-tail method. $\Delta\vec{r}_{net}$ stretches from Carolyn's initial position to her final position. The magnitude of her net displacement is found using the Pythagorean theorem:

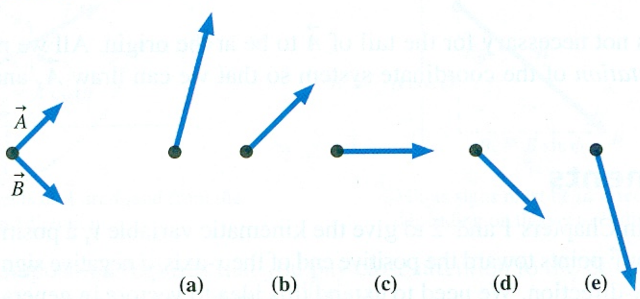
$$r_{net} = \sqrt{(120 \text{ km})^2 + (80 \text{ km})^2} = 144 \text{ km}$$

The direction of $\Delta\vec{r}_{net}$ is described by angle θ, which is

$$\theta = \tan^{-1}\left(\frac{80 \text{ km}}{120 \text{ km}}\right) = 34°$$

Thus Carolyn's net displacement is $\Delta\vec{r}_{net} = (144 \text{ km}, 34° \text{ north of east})$.

STOP TO THINK 3.2 Which figure shows $2\vec{A} - \vec{B}$?



3.3 Coordinate Systems and Vector Components

Vectors do not require a coordinate system. We can add and subtract vectors graphically, and we will do so frequently to clarify our understanding of a situation. But the graphical addition of vectors is not an especially good way to find quantitative results. In this section we will introduce a *coordinate representation* of vectors that will be the basis of an easier method for doing vector calculations.

Coordinate Systems

The world does not come with a coordinate system attached to it. A coordinate system is an artificially imposed grid that you place on a problem in order to make quantitative measurements. You are free to choose:

- Where to place the origin, and
- How to orient the axes.

Different problem solvers may choose to use different coordinate systems; that is perfectly acceptable. However, some coordinate systems will make a problem easier

A GPS uses satellite signals to find your position in the earth's coordinate system with amazing accuracy.

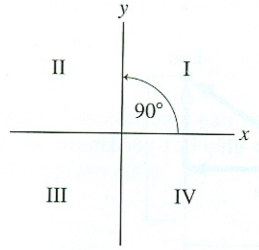


FIGURE 3.9 A conventional xy-coordinate system and the quadrants of the xy-plane.

to solve. Part of our goal is to learn how to choose an appropriate coordinate system for each problem.

FIGURE 3.9 shows the xy-coordinate system we will use in this book. The placement of the axes is not entirely arbitrary. By convention, the positive y-axis is located 90° *counterclockwise* (ccw) from the positive x-axis. Figure 3.9 also identifies the four **quadrants** of the coordinate system, I through IV.

Coordinate axes have a positive end and a negative end, separated by zero at the origin where the two axes cross. When you draw a coordinate system, it is important to label the axes. This is done by placing x and y labels at the *positive* ends of the axes, as in Figure 3.9. The purpose of the labels is twofold:

- To identify which axis is which, and
- To identify the positive ends of the axes.

This will be important when you need to determine whether the quantities in a problem should be assigned positive or negative values.

Component Vectors

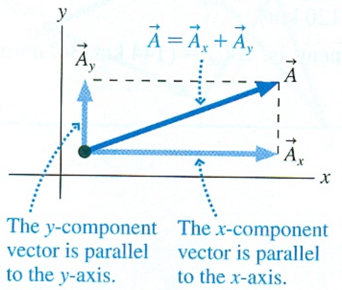


FIGURE 3.10 Component vectors $\vec{A}_x$ and $\vec{A}_y$ are drawn parallel to the coordinate axes such that $\vec{A} = \vec{A}_x + \vec{A}_y$.

FIGURE 3.10 shows a vector $\vec{A}$ and an xy-coordinate system that we've chosen. Once the directions of the axes are known, we can define two new vectors *parallel to the axes* that we call the **component vectors** of $\vec{A}$. You can see, using the parallelogram rule, that $\vec{A}$ is the vector sum of the two component vectors:

$$\vec{A} = \vec{A}_x + \vec{A}_y \tag{3.5}$$

In essence, we have broken vector $\vec{A}$ into two perpendicular vectors that are parallel to the coordinate axes. This process is called the **decomposition** of vector $\vec{A}$ into its component vectors.

NOTE It is not necessary for the tail of $\vec{A}$ to be at the origin. All we need to know is the *orientation* of the coordinate system so that we can draw $\vec{A}_x$ and $\vec{A}_y$ parallel to the axes.

Components

You learned in Chapters 1 and 2 to give the kinematic variable v_x a positive sign if the velocity vector $\vec{v}$ points toward the positive end of the x-axis, a negative sign if $\vec{v}$ points in the negative x-direction. We need to extend this idea to vectors in general.

Suppose vector $\vec{A}$ has been decomposed into component vectors $\vec{A}_x$ and $\vec{A}_y$ parallel to the coordinate axes. We can describe each component vector with a single number called the **component.** The *x-component* and *y-component* of vector $\vec{A}$, denoted A_x and A_y, are determined as follows:

TACTICS BOX 3.1 MP

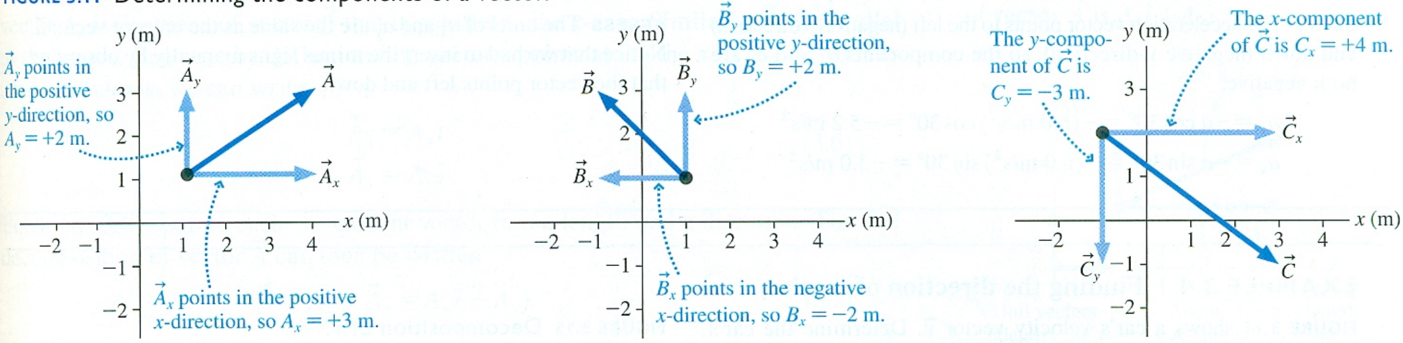
Determining the components of a vector

1. The absolute value $|A_x|$ of the x-component A_x is the magnitude of the component vector $\vec{A}_x$.
2. The sign of A_x is positive if $\vec{A}_x$ points in the positive x-direction (right), negative if $\vec{A}_x$ points in the negative x-direction (left).
3. The y-component A_y is determined similarly.

Exercises 10–18

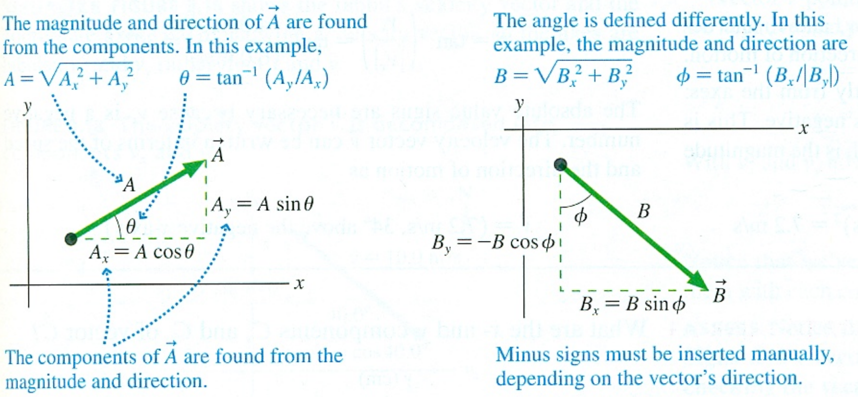
In other words, the component A_x tells us two things: how big $\vec{A}_x$ is and, with its sign, which end of the axis $\vec{A}_x$ points toward. **FIGURE 3.11** shows three examples of determining the components of a vector.

FIGURE 3.11 Determining the components of a vector.



NOTE Beware of the somewhat confusing terminology. $\vec{A}_x$ and $\vec{A}_y$ are called *component vectors,* whereas A_x and A_y are simply called *components.* The components A_x and A_y are just numbers (with units), so make sure you do *not* put arrow symbols over the components.

We will frequently need to decompose a vector into its components. We will also need to "reassemble" a vector from its components. In other words, we need to move back and forth between the geometric and the component representations of a vector. **FIGURE 3.12** shows how this is done.



◀ **FIGURE 3.12** Moving between the geometric representation and the component representation.

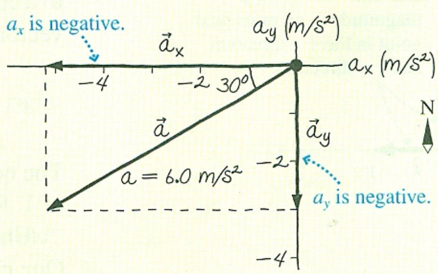
Each decomposition requires that you pay close attention to the direction in which the vector points and the angles that are defined.

- If a component vector points left (or down), you must *manually* insert a minus sign in front of the component, as was done for B_y in Figure 3.12.
- The role of sines and cosines can be reversed, depending upon which angle is used to define the direction. Compare A_x and B_x.
- The angle used to define direction is almost always between 0° and 90°, so you must take the inverse tangent of a positive number. Use absolute values of the components, as was done to find angle ϕ (Greek phi) in Figure 3.12.

EXAMPLE 3.3 Finding the components of an acceleration vector

Seen from above, a hummingbird's acceleration is (6.0 m/s^2, 30° south of west). Find the *x*- and *y*-components of the acceleration vector $\vec{a}$.

VISUALIZE It's important to *draw* vectors. **FIGURE 3.13** establishes a map-like coordinate system with the *x*-axis pointing east and the *y*-axis north. Vector $\vec{a}$ is then decomposed into components parallel to the axes. Notice that the axes are "acceleration axes" with units of acceleration, not *xy*-axes, because we're measuring an acceleration vector.



▶ **FIGURE 3.13** Decomposition of $\vec{a}$.

Continued

SOLVE The acceleration vector points to the left (negative x-direction) and down (negative y-direction), so the components a_x and a_y are both negative:

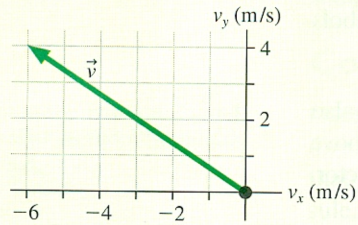
$$a_x = -a\cos 30° = -(6.0\ \text{m/s}^2)\cos 30° = -5.2\ \text{m/s}^2$$
$$a_y = -a\sin 30° = -(6.0\ \text{m/s}^2)\sin 30° = -3.0\ \text{m/s}^2$$

ASSESS The units of a_x and a_y are the same as the units of vector $\vec{a}$. Notice that we had to insert the minus signs manually by observing that the vector points left and down.

EXAMPLE 3.4 Finding the direction of motion

FIGURE 3.14 shows a car's velocity vector $\vec{v}$. Determine the car's speed and direction of motion.

FIGURE 3.14 The velocity vector $\vec{v}$ of Example 3.4.

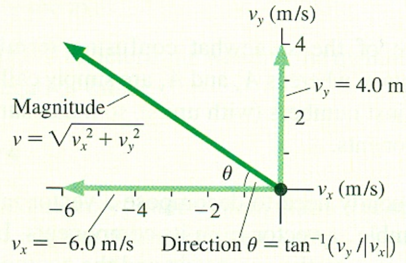


VISUALIZE **FIGURE 3.15** shows the components v_x and v_y and defines an angle θ with which we can specify the direction of motion.

SOLVE We can read the components of $\vec{v}$ directly from the axes: $v_x = -6.0$ m/s and $v_y = 4.0$ m/s. Notice that v_x is negative. This is enough information to find the car's speed v, which is the magnitude of $\vec{v}$:

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(-6.0\ \text{m/s})^2 + (4.0\ \text{m/s})^2} = 7.2\ \text{m/s}$$

FIGURE 3.15 Decomposition of $\vec{v}$.



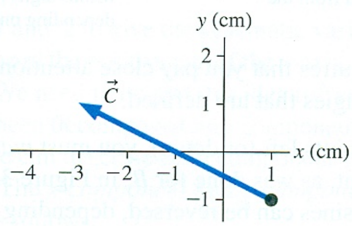
From trigonometry, angle θ is

$$\theta = \tan^{-1}\left(\frac{v_y}{|v_x|}\right) = \tan^{-1}\left(\frac{4.0\ \text{m/s}}{6.0\ \text{m/s}}\right) = 34°$$

The absolute value signs are necessary because v_x is a negative number. The velocity vector $\vec{v}$ can be written in terms of the speed and the direction of motion as

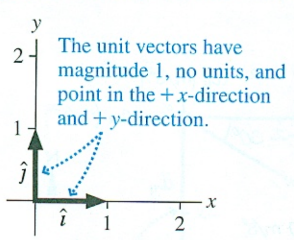
$$\vec{v} = (7.2\ \text{m/s},\ 34°\ \text{above the negative } x\text{-axis})$$

STOP TO THINK 3.3 What are the x- and y-components C_x and C_y of vector $\vec{C}$?



3.4 Unit Vectors and Vector Algebra

FIGURE 3.16 The unit vectors $\hat{\imath}$ and $\hat{\jmath}$.



The vectors (1, +x-direction) and (1, +y-direction), shown in **FIGURE 3.16**, have some interesting and useful properties. Each has a magnitude of 1, has no units, and is parallel to a coordinate axis. A vector with these properties is called a **unit vector.** These unit vectors have the special symbols

$$\hat{\imath} \equiv (1,\ \text{positive } x\text{-direction})$$
$$\hat{\jmath} \equiv (1,\ \text{positive } y\text{-direction})$$

The notation $\hat{\imath}$ (read "i hat") and $\hat{\jmath}$ (read "j hat") indicates a unit vector with a magnitude of 1. Recall that the symbol $\equiv$ means "is defined as."

Unit vectors establish the directions of the positive axes of the coordinate system. Our choice of a coordinate system may be arbitrary, but once we decide to place a coordinate system on a problem we need something to tell us "That direction is the positive x-direction." This is what the unit vectors do.

The unit vectors provide a useful way to write component vectors. The component vector $\vec{A}_x$ is the piece of vector $\vec{A}$ that is parallel to the x-axis. Similarly, $\vec{A}_y$ is parallel to the y-axis. Because, by definition, the vector $\hat{\imath}$ points along the x-axis and $\hat{\jmath}$ points along the y-axis, we can write

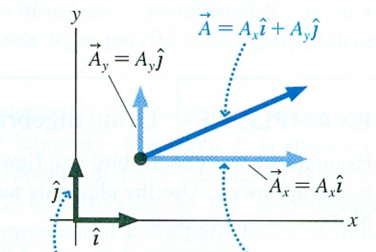
$$\vec{A}_x = A_x\hat{\imath}$$
$$\vec{A}_y = A_y\hat{\jmath} \tag{3.6}$$

Equations 3.6 separate each component vector into a length and a direction. The full decomposition of vector $\vec{A}$ can then be written

$$\vec{A} = \vec{A}_x + \vec{A}_y = A_x\hat{\imath} + A_y\hat{\jmath} \tag{3.7}$$

FIGURE 3.17 shows how the unit vectors and the components fit together to form vector $\vec{A}$.

FIGURE 3.17 The decomposition of vector $\vec{A}$ is $A_x\hat{\imath} + A_y\hat{\jmath}$.



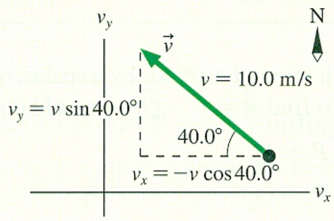
NOTE In three dimensions, the unit vector along the $+z$-direction is called $\hat{k}$, and to describe vector $\vec{A}$ we would include an additional component vector $\vec{A}_z = A_z\hat{k}$.

EXAMPLE 3.5 | **Run rabbit run!**

A rabbit, escaping a fox, runs 40.0° north of west at 10.0 m/s. A coordinate system is established with the positive x-axis to the east and the positive y-axis to the north. Write the rabbit's velocity in terms of components and unit vectors.

VISUALIZE FIGURE 3.18 shows the rabbit's velocity vector and the coordinate axes. We're showing a velocity vector, so the axes are labeled v_x and v_y rather than x and y.

FIGURE 3.18 The velocity vector $\vec{v}$ is decomposed into components v_x and v_y.



SOLVE 10.0 m/s is the rabbit's *speed,* not its velocity. The velocity, which includes directional information, is

$$\vec{v} = (10.0 \text{ m/s}, 40.0° \text{ north of west})$$

Vector $\vec{v}$ points to the left and up, so the components v_x and v_y are negative and positive, respectively. The components are

$$v_x = -(10.0 \text{ m/s}) \cos 40.0° = -7.66 \text{ m/s}$$

$$v_y = +(10.0 \text{ m/s}) \sin 40.0° = 6.43 \text{ m/s}$$

With v_x and v_y now known, the rabbit's velocity vector is

$$\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} = (-7.66\hat{\imath} + 6.43\hat{\jmath}) \text{ m/s}$$

Notice that we've pulled the units to the end, rather than writing them with each component.

ASSESS Notice that the minus sign for v_x was inserted manually. **Signs don't occur automatically; you have to set them after checking the vector's direction.**

Vector Math

You learned in Section 3.2 how to add vectors graphically, but it is a tedious problem in geometry and trigonometry to find precise values for the magnitude and direction of the resultant. The addition and subtraction of vectors become much easier if we use components and unit vectors.

To see this, let's evaluate the vector sum $\vec{D} = \vec{A} + \vec{B} + \vec{C}$. To begin, write this sum in terms of the components of each vector:

$$\vec{D} = D_x\hat{\imath} + D_y\hat{\jmath} = \vec{A} + \vec{B} + \vec{C}$$
$$= (A_x\hat{\imath} + A_y\hat{\jmath}) + (B_x\hat{\imath} + B_y\hat{\jmath}) + (C_x\hat{\imath} + C_y\hat{\jmath}) \tag{3.8}$$

We can group together all the x-components and all the y-components on the right side, in which case Equation 3.8 is

$$(D_x)\hat{\imath} + (D_y)\hat{\jmath} = (A_x + B_x + C_x)\hat{\imath} + (A_y + B_y + C_y)\hat{\jmath} \tag{3.9}$$

Comparing the x- and y-components on the left and right sides of Equation 3.9, we find:

$$D_x = A_x + B_x + C_x$$
$$D_y = A_y + B_y + C_y \tag{3.10}$$

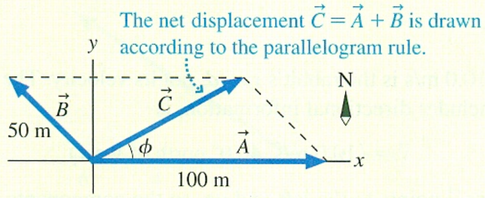
Stated in words, Equation 3.10 says that we can perform vector addition by adding the x-components of the individual vectors to give the x-component of the resultant and by adding the y-components of the individual vectors to give the y-component of the resultant. This method of vector addition is called **algebraic addition.**

EXAMPLE 3.6 Using algebraic addition to find a displacement

Example 3.1 was about a bird that flew 100 m to the east, then 50 m to the northwest. Use the algebraic addition of vectors to find the bird's net displacement.

VISUALIZE FIGURE 3.19 shows displacement vectors $\vec{A} = (100\text{ m, east})$ and $\vec{B} = (50\text{ m, northwest})$. We draw vectors tip-to-tail to add them graphically, but it's usually easier to draw them all from the origin if we are going to use algebraic addition.

FIGURE 3.19 The net displacement is $\vec{C} = \vec{A} + \vec{B}$.



SOLVE To add the vectors algebraically we must know their components. From the figure these are seen to be

$$\vec{A} = 100\,\hat{\imath}\text{ m}$$
$$\vec{B} = (-50\cos 45°\,\hat{\imath} + 50\sin 45°\hat{\jmath})\text{ m} = (-35.3\hat{\imath} + 35.3\hat{\jmath})\text{ m}$$

Notice that vector quantities must include units. Also notice, as you would expect from the figure, that $\vec{B}$ has a negative x-component. Adding $\vec{A}$ and $\vec{B}$ by components gives

$$\begin{aligned}\vec{C} &= \vec{A} + \vec{B} = 100\hat{\imath}\text{ m} + (-35.3\hat{\imath} + 35.3\hat{\jmath})\text{ m}\\ &= (100\text{ m} - 35.3\text{ m})\hat{\imath} + (35.3\text{ m})\hat{\jmath} = (64.7\hat{\imath} + 35.3\hat{\jmath})\text{ m}\end{aligned}$$

This would be a perfectly acceptable answer for many purposes. However, we need to calculate the magnitude and direction of $\vec{C}$ if we want to compare this result to our earlier answer. The magnitude of $\vec{C}$ is

$$C = \sqrt{C_x^2 + C_y^2} = \sqrt{(64.7\text{ m})^2 + (35.3\text{ m})^2} = 74\text{ m}$$

The angle ϕ, as defined in Figure 3.19, is

$$\phi = \tan^{-1}\left(\frac{C_y}{|C_x|}\right) = \tan^{-1}\left(\frac{35.3\text{ m}}{64.7\text{ m}}\right) = 29°$$

Thus $\vec{C} = (74\text{ m, }29°\text{ north of west})$, in perfect agreement with Example 3.1.

Vector subtraction and the multiplication of a vector by a scalar, using components, are very much like vector addition. To find $\vec{R} = \vec{P} - \vec{Q}$ we would compute

$$\begin{aligned}R_x &= P_x - Q_x\\ R_y &= P_y - Q_y\end{aligned} \tag{3.11}$$

Similarly, $\vec{T} = c\vec{S}$ would be

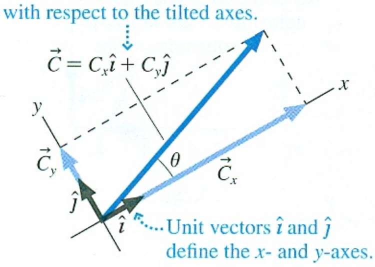
$$\begin{aligned}T_x &= cS_x\\ T_y &= cS_y\end{aligned} \tag{3.12}$$

In other words, a vector equation is interpreted as meaning: Equate the x-components on both sides of the equals sign, then equate the y-components, and then the z-components. Vector notation allows us to write these three equations in a much more compact form.

Tilted Axes and Arbitrary Directions

As we've noted, the coordinate system is entirely your choice. It is a grid that you impose on the problem in a manner that will make the problem easiest to solve. As you've already seen in Chapter 2, it is often convenient to tilt the axes of the coordinate system, such as those shown in FIGURE 3.20. The axes are perpendicular, and the y-axis is oriented correctly with respect to the x-axis, so this is a legitimate coordinate system. There is no requirement that the x-axis has to be horizontal.

FIGURE 3.20 A coordinate system with tilted axes.



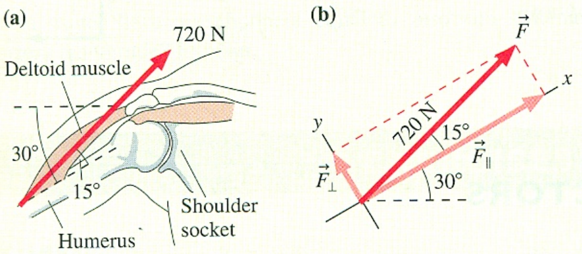
Finding components with tilted axes is no harder than what we have done so far. Vector $\vec{C}$ in Figure 3.20 can be decomposed into $\vec{C} = C_x\hat{\imath} + C_y\hat{\jmath}$, where $C_x = C\cos\theta$ and $C_y = C\sin\theta$. Note that the unit vectors $\hat{\imath}$ and $\hat{\jmath}$ correspond to the *axes*, not to "horizontal" and "vertical," so they are also tilted.

Tilted axes are useful if you need to determine component vectors "parallel to" and "perpendicular to" an arbitrary line or surface. This is illustrated in the following example.

EXAMPLE 3.7 Muscle and bone

The deltoid—the rounded muscle across the top of your upper arm—allows you to lift your arm away from your side. It does so by pulling on an attachment point on the humerus, the upper arm bone, at an angle of 15° with respect to the humerus. If you hold your arm at an angle 30° below horizontal, the deltoid must pull with a force of 720 N to support the weight of your arm, as shown in FIGURE 3.21a. (You'll learn in Chapter 5 that force is a vector quantity measured in units of *newtons*, abbreviated N.) What are the components of the muscle force parallel to and perpendicular to the bone?

FIGURE 3.21 Finding the components of force parallel and perpendicular to the humerus.



VISUALIZE FIGURE 3.21b shows a tilted coordinate system with the x-axis parallel to the humerus. The force $\vec{F}$ is shown 15° from the x-axis. The component of force parallel to the bone, which we can denote $F_{\parallel}$, is equivalent to the x-component: $F_{\parallel} = F_x$. Similarly, the component of force perpendicular to the bone is $F_{\perp} = F_y$.

SOLVE From the geometry of Figure 3.21b, we see that

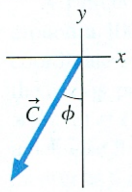
$$F_{\parallel} = F\cos 15° = (720\text{ N})\cos 15° = 695\text{ N}$$

$$F_{\perp} = F\sin 15° = (720\text{ N})\sin 15° = 186\text{ N}$$

ASSESS The muscle pulls nearly parallel to the bone, so we expected $F_{\parallel} \approx 720$ N and $F_{\perp} \ll F_{\parallel}$. Thus our results seem reasonable.

STOP TO THINK 3.4 Angle ϕ that specifies the direction of $\vec{C}$ is given by

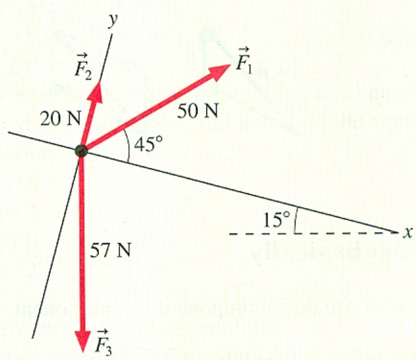
a. $\tan^{-1}(|C_x|/C_y)$
b. $\tan^{-1}(C_x/|C_y|)$
c. $\tan^{-1}(|C_x|/|C_y|)$
d. $\tan^{-1}(|C_y|/C_x)$
e. $\tan^{-1}(C_y/|C_x|)$
f. $\tan^{-1}(|C_y|/|C_x|)$



CHALLENGE EXAMPLE 3.8 Finding the net force

FIGURE 3.22 shows three forces acting at one point. What is the net force $\vec{F}_{\text{net}} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$?

FIGURE 3.22 Three forces.



VISUALIZE Figure 3.22 shows the forces and a tilted coordinate system.

SOLVE The vector equation $\vec{F}_{\text{net}} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$ is really two simultaneous equations:

$$(F_{\text{net}})_x = F_{1x} + F_{2x} + F_{3x}$$
$$(F_{\text{net}})_y = F_{1y} + F_{2y} + F_{3y}$$

The components of the forces are determined with respect to the axes. Thus

$$F_{1x} = F_1\cos 45° = (50\text{ N})\cos 45° = 35\text{ N}$$
$$F_{1y} = F_1\sin 45° = (50\text{ N})\sin 45° = 35\text{ N}$$

$\vec{F}_2$ is easier. It is pointing along the y-axis, so $F_{2x} = 0$ N and $F_{2y} = 20$ N. To find the components of $\vec{F}_3$, we need to recognize—because $\vec{F}_3$ points straight down—that the angle between $\vec{F}_3$ and the x-axis is 75°. Thus

$$F_{3x} = F_3\cos 75° = (57\text{ N})\cos 75° = 15\text{ N}$$
$$F_{3y} = -F_3\sin 75° = -(57\text{ N})\sin 75° = -55\text{ N}$$

The minus sign in F_{3y} is critical, and it appears not from some formula but because we recognized—from the figure—that the y-component of $\vec{F}_3$, points in the $-y$-direction. Combining the pieces, we have

$$(F_{\text{net}})_x = 35\text{ N} + 0\text{ N} + 15\text{ N} = 50\text{ N}$$
$$(F_{\text{net}})_y = 35\text{ N} + 20\text{ N} + (-55\text{ N}) = 0\text{ N}$$

Thus the net force is $\vec{F}_{\text{net}} = 50\hat{\imath}$ N. It points along the x-axis of the tilted coordinate system.

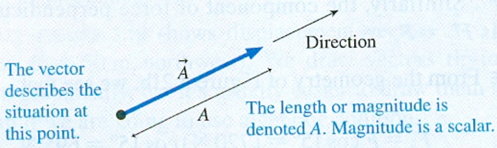
ASSESS Notice that all work was done with reference to the axes of the coordinate system, not with respect to vertical or horizontal.

SUMMARY

The goals of Chapter 3 have been to learn how vectors are represented and used.

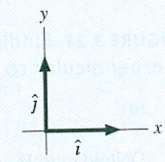
IMPORTANT CONCEPTS

A vector is a quantity described by both a magnitude and a direction.



Unit Vectors

Unit vectors have magnitude 1 and no units. Unit vectors $\hat{\imath}$ and $\hat{\jmath}$ define the directions of the x- and y-axes.



USING VECTORS

Components

The component vectors are parallel to the x- and y-axes:

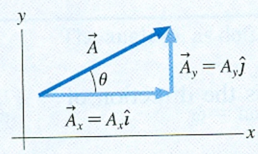
$$\vec{A} = \vec{A}_x + \vec{A}_y = A_x\hat{\imath} + A_y\hat{\jmath}$$

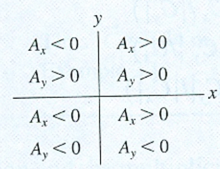
In the figure at the right, for example:

$$A_x = A\cos\theta \qquad A = \sqrt{A_x^2 + A_y^2}$$

$$A_y = A\sin\theta \qquad \theta = \tan^{-1}(A_y/A_x)$$

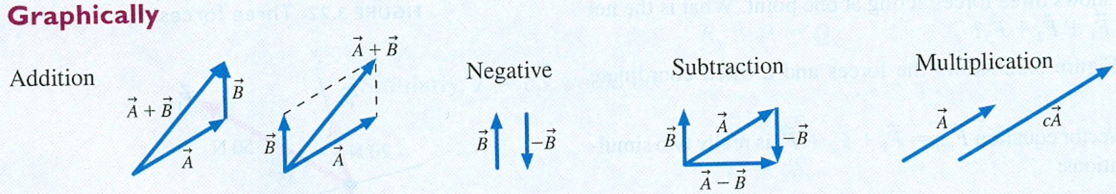
► Minus signs need to be included if the vector points down or left.





The components A_x and A_y are the magnitudes of the component vectors $\vec{A}_x$ and $\vec{A}_y$ *and* a plus or minus sign to show whether the component vector points toward the positive end or the negative end of the axis.

Working Graphically



Working Algebraically

Vector calculations are done component by component: $\vec{C} = 2\vec{A} + \vec{B}$ means $\begin{cases} C_x = 2A_x + B_x \\ C_y = 2A_y + B_y \end{cases}$

The magnitude of $\vec{C}$ is then $C = \sqrt{C_x^2 + C_y^2}$ and its direction is found using $\tan^{-1}$.

TERMS AND NOTATION

scalar
vector
magnitude
resultant vector
graphical addition
zero vector, $\vec{0}$
quadrants
component vector
decomposition
component
unit vector, $\hat{\imath}$ or $\hat{\jmath}$
algebraic addition