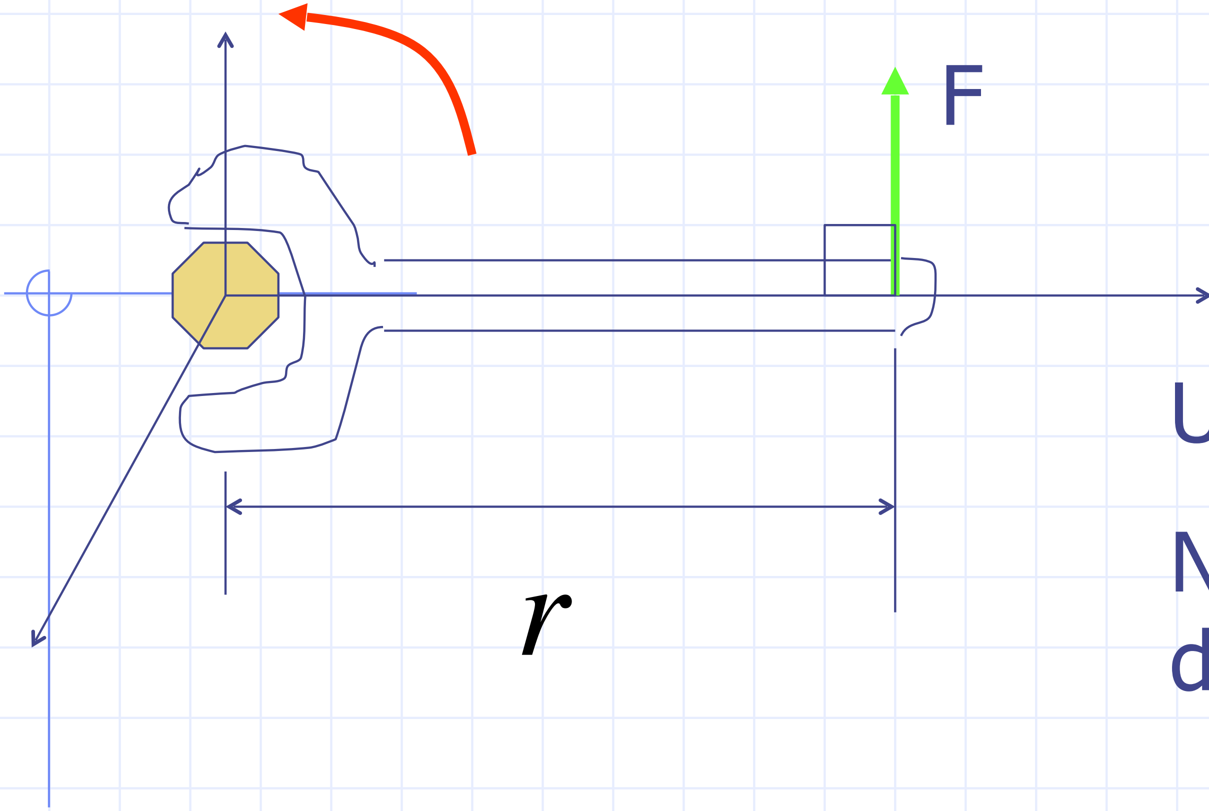
Chapter 9. Rotational Dynamics

- As we did for linear (or translational) motion, we studied kinematics (motion without regard to the cause) and then dynamics (motion with regard to the cause), we now proceed in a similar fashion
- We know that forces are responsible for linear motion
- We will now see that rotational motion is caused by torques
- Consider a wrench of length L=r



Torque

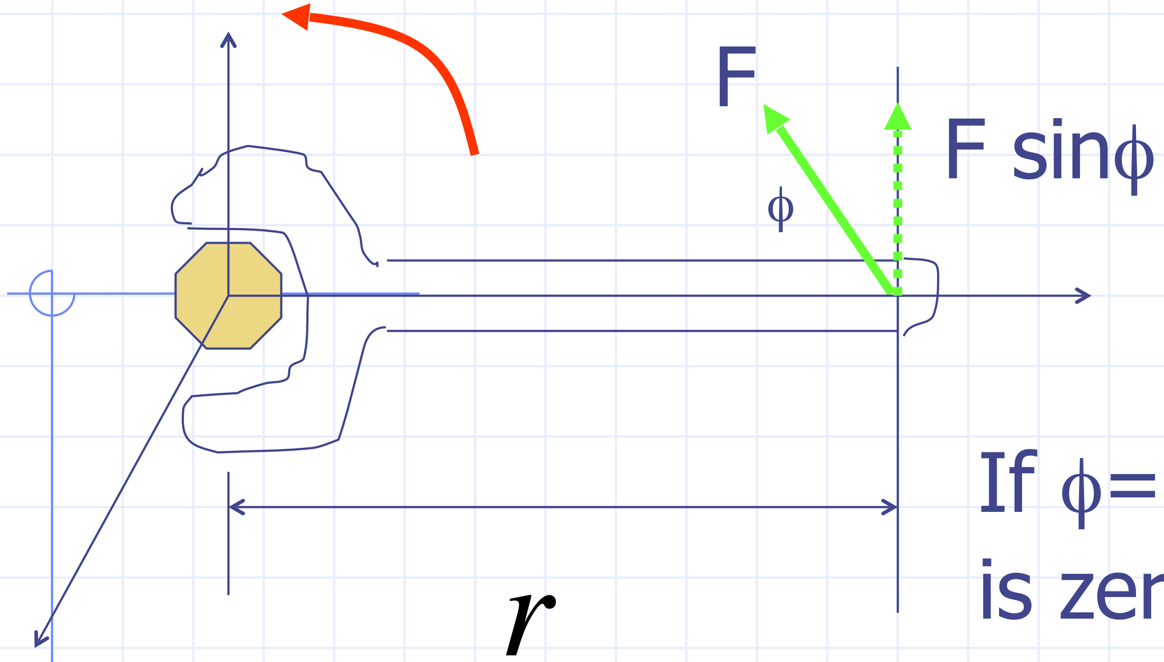
$$\tau = rF$$

Units of N m

Not work or energy, do not use Joules

r is called the lever arm. Must always be perpendicular to the force

If the force is not perpendicular to the level arm, we need to find the component that is perpendicular (either the force or the lever arm)



$$\tau = rF\sin\phi$$

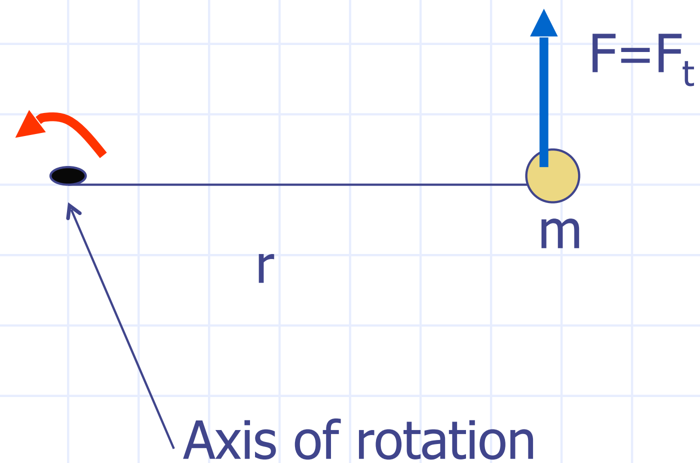
If ϕ=0, then the torque is zero

- Therefore, torques (force times length) are responsible for rotational motion

(really from Section 11.4)

Newton's 2nd Law for Rotational Motion

- The torque for a point-particle of mass m a distance r from the rotation axis is



$$\tau = rF_t$$

$$F_t = ma_{t,}\; a_t = r\alpha$$

$$\Rightarrow \tau = ma_t r = mr^2\alpha$$

- Define I = mr^2 = Moment of Inertia for a point particle; a scalar, units of kg m^2
- A rigid body is composed of many, many particles

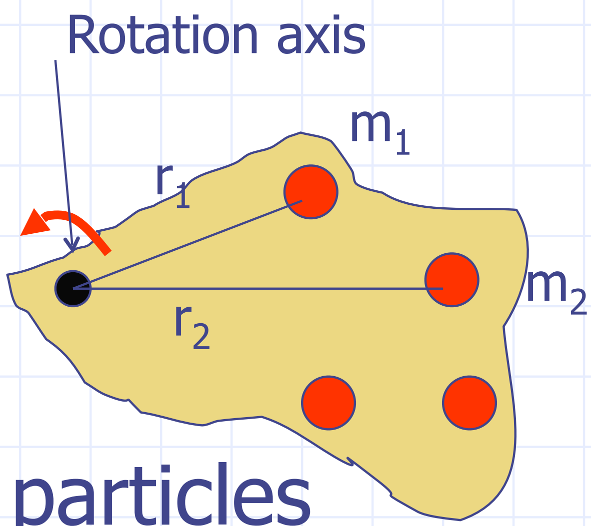
of mass m_i which are r_i from the axis of rotation

- Each of these masses creates a torque about the axis of rotation

$$\tau_1 = m_1 r_1^2 \alpha$$

$$\tau_2 = m_2 r_2^2 \alpha$$

....



- Sum up all torques due to all particles

$$\sum \tau_i = \sum m_i r_i^2 \alpha$$

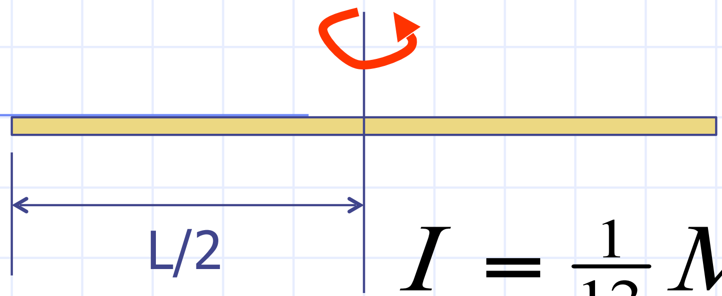
α is the same for all particles

$$I = \sum m_i r_i^2$$

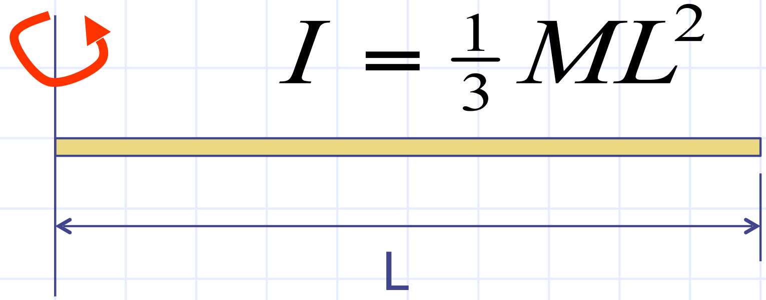
$$\sum \tau = I\alpha$$

I =moment of inertia for the rigid body. It is different for different shaped objects and for different axes of rotation. Fig. 9.18.

- For a thin rod of mass M and length L



$$I = \tfrac{1}{12} ML^2$$



$$I = \tfrac{1}{3} ML^2$$

- The last equation is Newton's 2nd Law for rotation. Compare to the translational form

$$\sum \vec{\tau} = I\vec{\alpha} \quad \Leftrightarrow \quad \sum \vec{F} = m\vec{a}$$

Example Problem

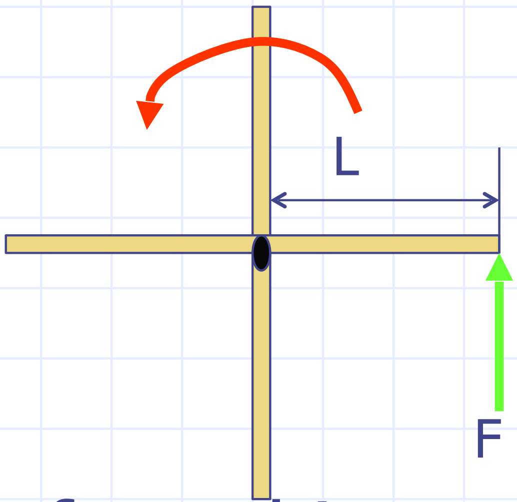
A rotating door is made of 4 rectangular panes each with a mass of 85 kg. A person pushes on the outer edge of one pane with a force of 68 N, directed perpendicular to the pane. Determine the door's α.

Given: L = 1.2 m, m_{pane}=85 kg,

F=68 N

$$\sum \tau = I\alpha$$

$$FL = I\alpha \Rightarrow \alpha = \frac{FL}{I}$$



From Fig. 9.18, moment of inertia for a thin rectangular rod is (same as a thin sheet)

Since there are 4 panes.

$$I_{pane} = \tfrac{1}{3}ML^2 \Rightarrow I_{door} = \tfrac{4}{3}ML^2$$

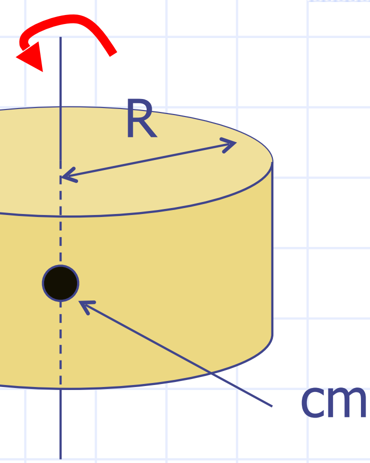
$$\alpha = \frac{FL}{I} = \frac{FL}{\tfrac{4}{3}ML^2}$$

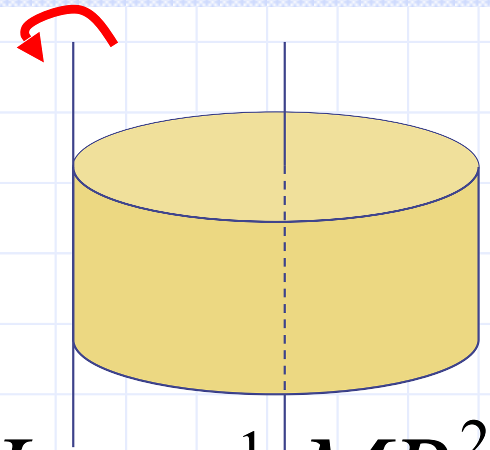
$$\alpha = \frac{3F}{4ML} = \frac{3(68\text{ N})}{4(85\text{ kg})(1.2\text{ m})} = 0.50\,\frac{\text{N}}{\text{kg m}} = 0.50\,\frac{\text{rad}}{\text{s}^2}$$

Example Problem

The parallel axis theorem provides a useful way to calculate I about an arbitrary axis. The theorem states that $I = I_{cm} + MD^2$, where I_{cm} is the moment of inertia of an object (of mass M) with an axis that passes through the center of mass and is parallel to the axis of interest. D is the perpendicular distance between the two axes. Now, determine I of a solid cylinder of radius R for an axis that lies on the surface of the cylinder and perpendicular to the circular ends.

Solution: The center of mass of the cylinder is on a line defining the axis of the cylinder





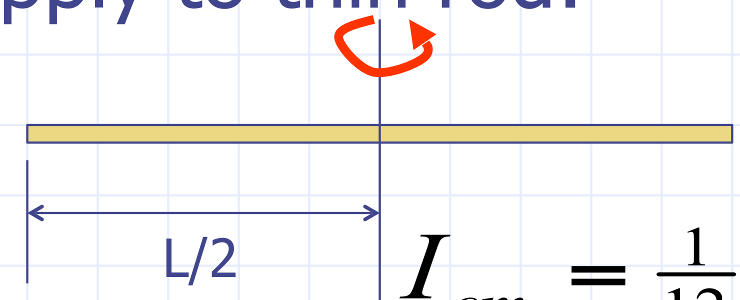
From Fig. 9.18:

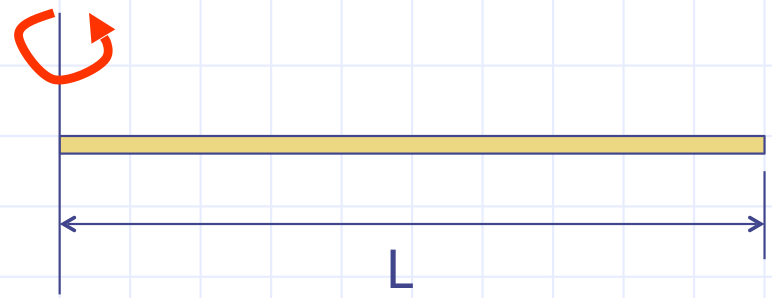
$$I_{cm} = \tfrac{1}{2} MR^2$$

From the parallel axis theorem with D=R:

$$I = I_{cm} + MD^2 = \tfrac{1}{2} MR^2 + MR^2 = \boxed{\tfrac{3}{2} MR^2}$$

Apply to thin rod:





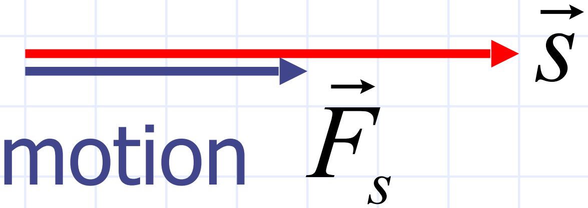
$$I_{cm} = \tfrac{1}{12} ML^2$$

$$I = I_{cm} + MD^2 = \tfrac{1}{12} ML^2 + M\left(\frac{L}{2}\right)^2 = \boxed{\tfrac{1}{3} ML^2}$$

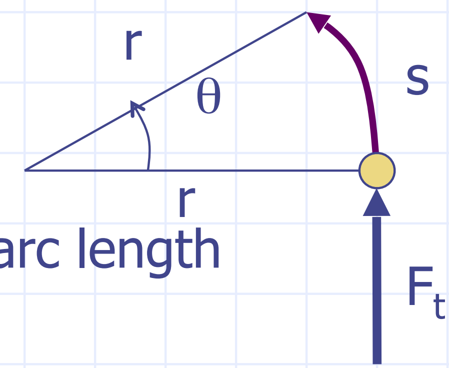
Rotational Work

- For translational motion, we defined the work as

$$W = F_s s$$



- For rotational motion



s = arc length

= rθ

$$W = F_s s = F_t r\theta = (F_t r)\theta$$

$$\boxed{W_R = \tau\,\theta}$$

Units of N m or J when θ is in radians

Rotational Kinetic Energy

- For translational motion, the K was defined

$$K = \tfrac{1}{2} m\text{v}^2, \quad \textbf{since } \text{v}_t = r\omega$$

$$K = \tfrac{1}{2} m(r\omega)^2 = \tfrac{1}{2} mr^2\omega^2$$

- For a point particle $I = mr^2$, therefore

$$K_{rot} = \tfrac{1}{2} I\omega^2$$

- Or for a rigid body

$$K_{rot} = \tfrac{1}{2} \sum m_i r_i^2 \omega^2 = \boxed{\tfrac{1}{2} I\omega^2 = K_{rot}}$$

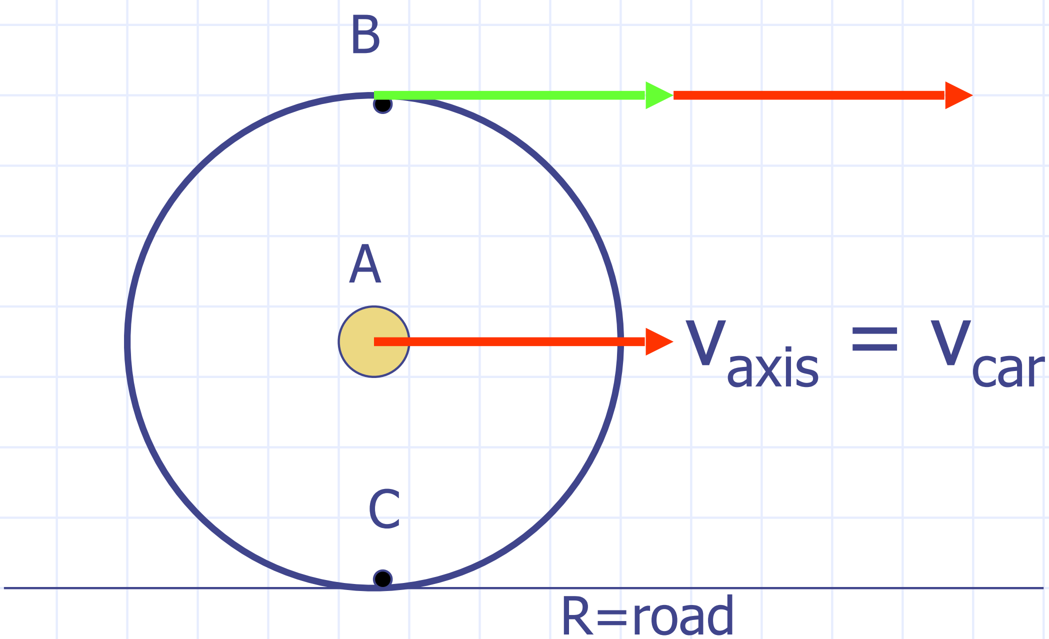
- For a rigid body that has both translational and rotational motion, its total kinetic energy is

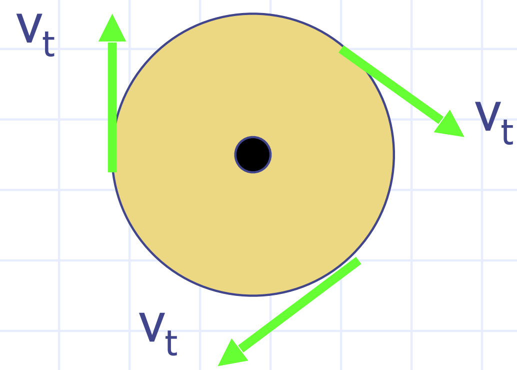
$$K_{total} = K + K_{rot} = \tfrac{1}{2} m\text{v}^2 + \tfrac{1}{2} I\omega^2$$

- The total mechanical energy is then

$$E_{total} = \tfrac{1}{2} m\text{v}^2 + \tfrac{1}{2} I\omega^2 + mgy + \tfrac{1}{2} kx^2$$

Rolling Motion (Tires, Billiards)





If tire is suspended, every point on edge has same v_t

- Consider a tire traveling on a road with friction (no skidding) between the tire and road
- First, review concept of relative velocity. What is the velocity of B as seen by the ground?

- Point B has a velocity v_t with respect to the A
- Now, A (the tire as a whole) is moving to the right with velocity v_{car}

$$\mathbf{v}_{B,A} = \mathbf{v}_{t}, \quad \mathbf{v}_{A,G} = \mathbf{v}_{car}$$

$$\mathbf{v}_{B,G} = \mathbf{v}_{B,A} + \mathbf{v}_{A,G}, \quad \mathbf{v}_{t} = \mathbf{v}_{car}$$

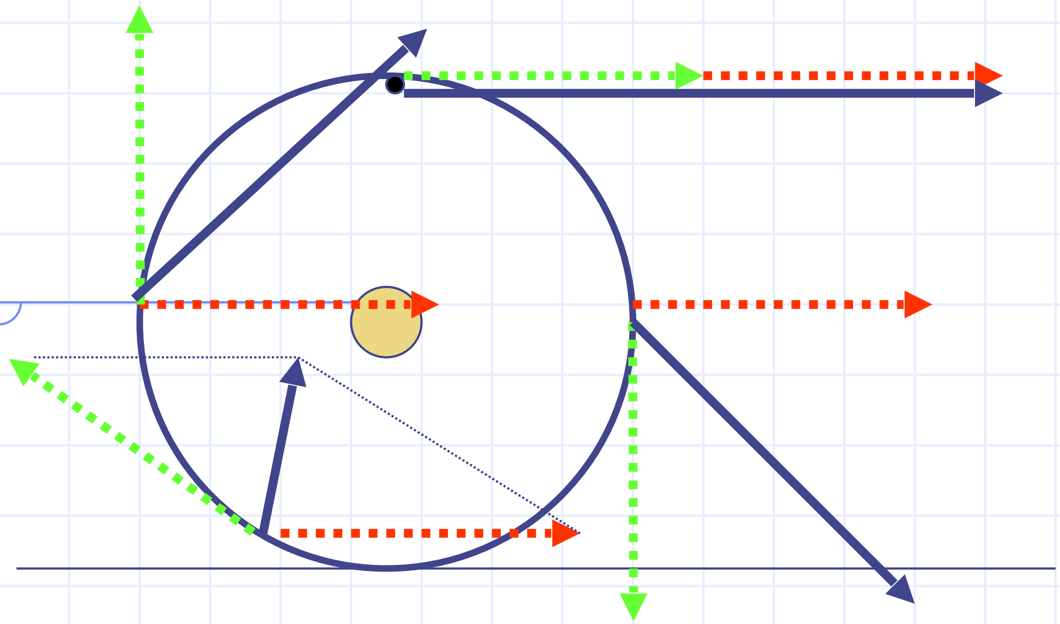
$$\mathbf{v}_{B,G} = \mathbf{v}_{t} + \mathbf{v}_{car} = 2\mathbf{v}_{car}$$

- What is velocity of C as seen by the ground?

$$\mathbf{v}_{C,A} = \mathbf{v}_{t}, \quad \mathbf{v}_{A,G} = \mathbf{v}_{car}$$

$$\mathbf{v}_{C,G} = \mathbf{v}_{C,A} + \mathbf{v}_{A,G}, \quad \mathbf{v}_{t} = -\mathbf{v}_{car}$$

$$\mathbf{v}_{C,G} = \mathbf{v}_{t} + \mathbf{v}_{car} = 0$$



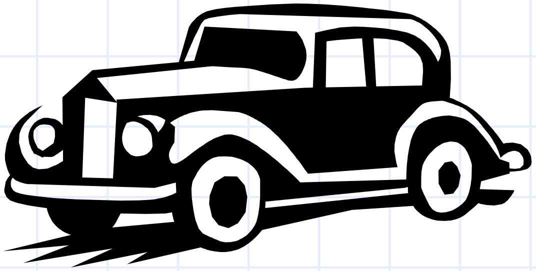
Example Problem

A tire has a radius of 0.330 m, and its center moves forward with a linear speed of 15.0 m/s. (a) What is ω of the wheel? (b) Relative to the axle, what is v_t of a point located 0.175 m from the axle?

$$\omega = \frac{\mathbf{v}_t}{r} = \frac{\mathbf{v}_{car}}{r} = \frac{15.0}{0.330} = \boxed{45.5\ \mathbf{rad/s}}$$

$$\mathbf{v}_t = r\omega = (0.175)(\mathbf{45.5}) = \boxed{\mathbf{7.96\ m/s}}$$

Example



A car is moving with a speed of 27.0 m/s. Each wheel has a radius of 0.300 m and a moment of inertia of 0.850 kg m^2. The car has a total mass (including the wheels) of 1.20×10^3 kg. Find (a) the translational K of the entire car, (b) the total K_{rot} of the four wheels, and (c) the total K of the car.

<u>Solution:</u>

Given: $v_{car} = 27.0$ m/s, $m_{car} = 1.20 \times 10^3$ kg, $r_w = 0.300$ m, $I_w = 0.850$ kg m^2

a)

$$K = \tfrac{1}{2} m_{car} v_{car}^2 = \tfrac{1}{2}(1.2\text{x}10^3)(27.0)^2$$

$$K = 4.37\text{x}10^5 \text{ J}$$

b)

$$K_{rot} = \tfrac{1}{2} I\omega^2 = \tfrac{1}{2} I_w \left(\frac{v_t}{r_w}\right)^2 = \tfrac{1}{2} I_w \left(\frac{v_{car}}{r_w}\right)^2$$

$$K_{rot} = \tfrac{1}{2}(0.850)(27.0/0.300)^2$$

$$= 3.44\text{x}10^3 \text{ J}$$

$$K_{wheels} = 4K_{rot} = 1.38\text{x}10^4 \text{ J}$$

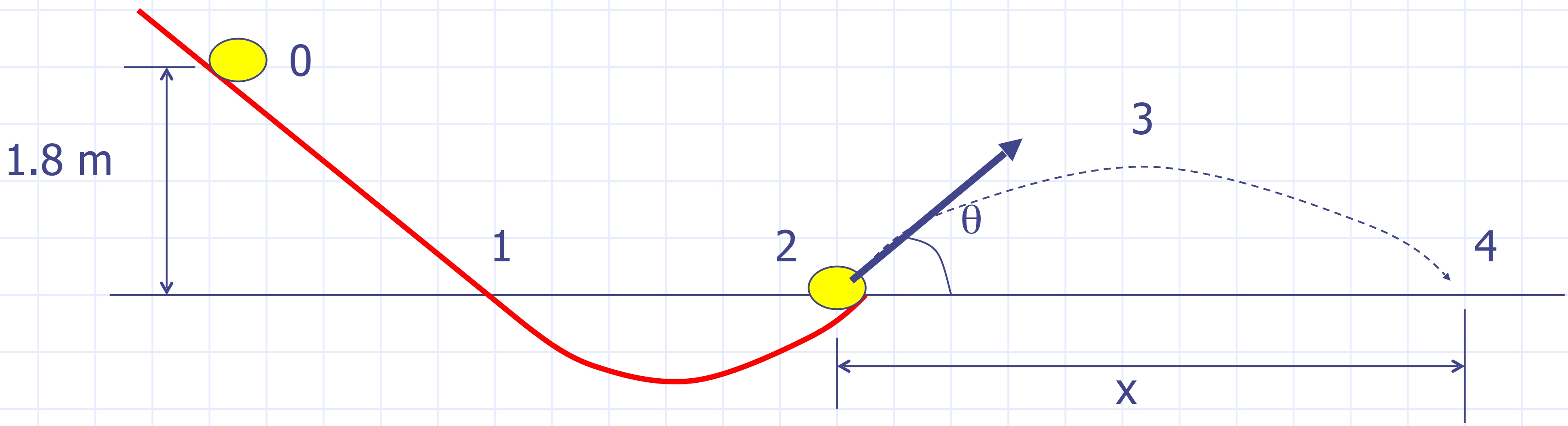
c)

$$K_{total} = K + K_{wheels}$$

$$= 4.37\text{x}10^5 + 1.38\text{x}10^4 = 4.51\text{x}10^5 \text{ J}$$

Example Problem

A tennis ball, starting from rest, rolls down a hill into a valley. At the top of the valley, the ball becomes airborne, leaving at an angle of 35° with respect to the horizontal. Treat the ball as a thin-walled spherical shell and determine the horizontal distance the ball travels after becoming airborne.



Solution:

Given: $v_0 = 0$, $\omega_0 = 0$, $y_0 = 1.8$ m $= h$, $y_1 = y_2 = y_4 = 0$, $\theta_2 = 35°$, $x_2 = 0$, $I=(2/3)MR^2$

Find: x_4 ?

Method: As there is no "friction" or air resistance in the problem, therefore no non-conservative forces, we can use conservation of mechanical energy

$$E_{total} = \tfrac{1}{2}m\mathrm{v}^2 + \tfrac{1}{2}I\omega^2 + mgy$$

$$E_0 = mgh$$

$$E_1 = \tfrac{1}{2}m\mathrm{v}_1^2 + \tfrac{1}{2}I\omega_1^2 + mg(0)$$

$$E_2 = \tfrac{1}{2}m\mathrm{v}_2^2 + \tfrac{1}{2}I\omega_2^2 + mg(0) = E_1$$

$$\Rightarrow v_1 = v_2,\ \omega_1 = \omega_2$$

Since $E_0 = E_1 = E_2$

$$mgh = \tfrac{1}{2}mv_2^2 + \tfrac{1}{2}I\omega_2^2,\ \text{but } v_t = r\omega$$

$$v_t = v_2$$

Velocity of ball equals tangential velocity at edge of ball

$$mgh = \tfrac{1}{2}mv_2^2 + \tfrac{1}{2}(\tfrac{2}{3}mr^2)(v_2/r)^2$$

$$gh = \tfrac{1}{2}v_2^2 + \tfrac{1}{3}v_2^2 = \tfrac{5}{6}v_2^2$$

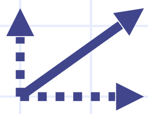
$$\Rightarrow \boxed{v_2 = \sqrt{6gh/5}}$$

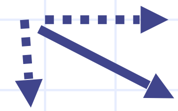
$$= \sqrt{6(9.80)(1.8)/5} = 4.6\ \text{m/s}$$

Now, use 2D kinematic equations for projectile motion

$$v_{2x} = v_2 \cos\theta_2 = v_{4x}$$

$$v_{2y} = v_2 \sin\theta_2 = -v_{4y}$$





$$v_{4y} = v_{2y} - g(t_4 - t_2),\ t_2 = 0$$

$$-v_{2y} = v_{2y} - gt_4$$

$$-2v_{2y} = -gt_4 \Rightarrow \boxed{t_4 = 2v_{2y} / g}$$

$$x_4 = x_2 + \tfrac{1}{2}(v_{2x} + v_{4x})t_4 = v_{2x}t_4$$

$$x_4 = v_{2x} 2v_{2y} / g = 2v_2 \cos\theta_2 v_2 \sin\theta_2 / g$$

$$x_4 = \frac{2v_2^2 \cos\theta_2 \sin\theta_2}{g} = \frac{6gh}{5} \frac{2\cos\theta_2 \sin\theta_2}{g}$$

$$\boxed{x_4 = \frac{12h\cos\theta_2 \sin\theta_2}{5}} = \frac{12(1.8)\cos 35 \sin 35}{5}$$

$$\boxed{x_4 = 2.0 \text{ m}}$$

Example Problem

- A mass m_1 (=15.0 kg) and a mass m_2 (=10.0 kg) are suspended by a pulley that has a radius R (=10.0 cm) and a mass M (=3.00 kg). The cord has negligible mass and causes the pulley to rotate without slipping. The pulley rotates about its axis without friction. The masses start from rest at a distance h (=3.00 m) part. Treating the pulley as a uniform disk, determine the speeds of the two masses as they pass each other.