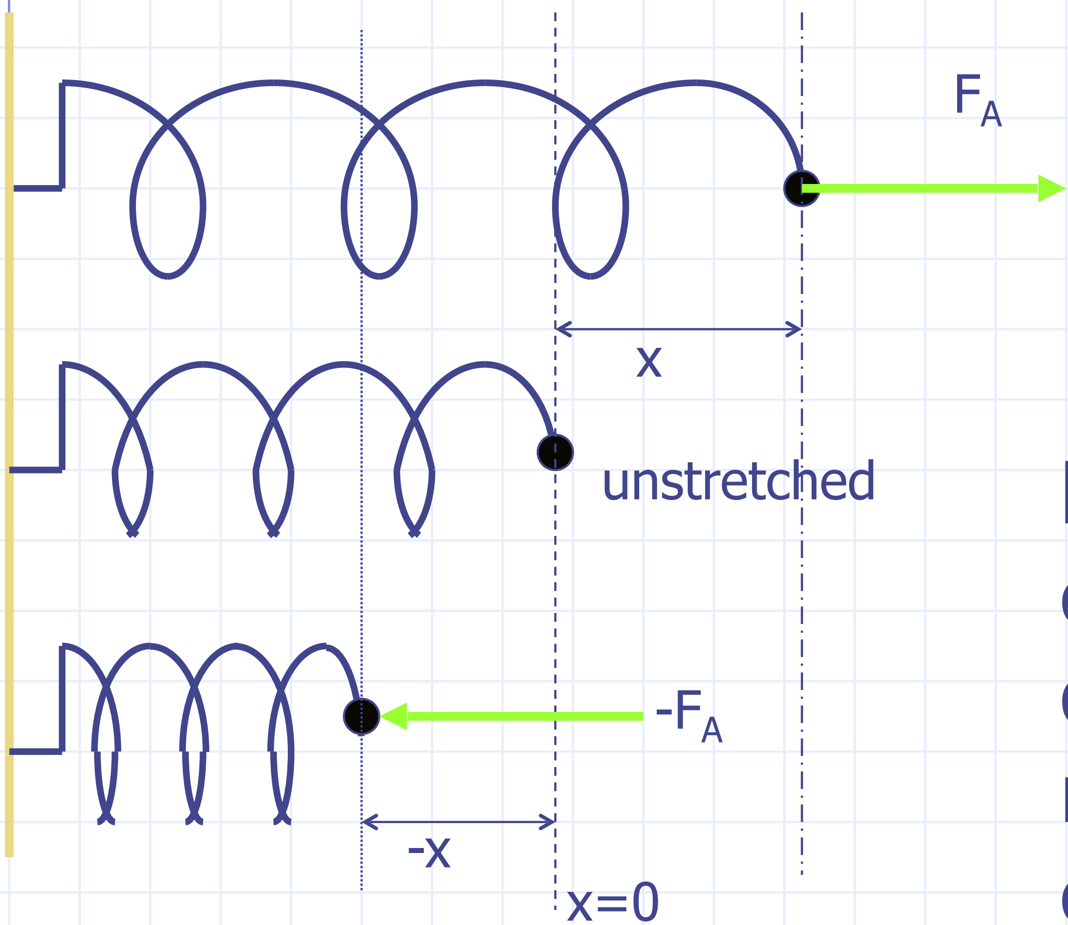
The Spring

- Consider a spring, which we apply a force F_A to either stretch it or compress it



$$F_A = kx$$

k is the spring constant, units of N/m, different for different materials, number of coils

From Newton's 3rd Law, the spring exerts a force that is equal in magnitude, but opposite in direction

$$F_s = -kx$$

Hooke's Law for the restoring force of an ideal spring. (It is a conservative force.)

Chapter 4, problem P29

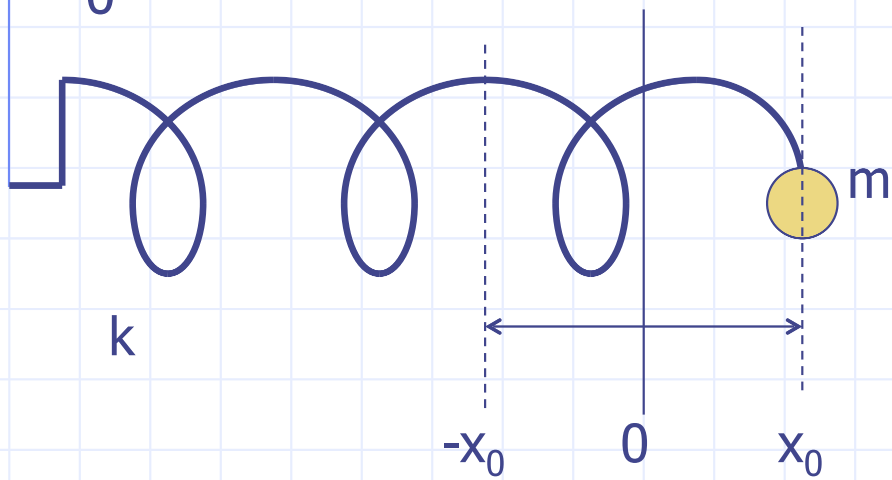
Five identical springs, each with stiffness 390 N/m, are attached in parallel (that is side-by-side) to hold up a heavy weight. If these springs were replaced by an equivalent single spring, what should be the stiffness of this single spring?

Oscillatory Motion

- We continue our studies of mechanics, but combine the concepts of translational and rotational motion.
- In particular, we will re-examine the restoring force of the spring (later its potential energy).
- We will consider the motion of a mass, attached to the spring, about its equilibrium position.
- This type of motion is applicable to many other kinds of situations: pendulum, atoms, planets, ...

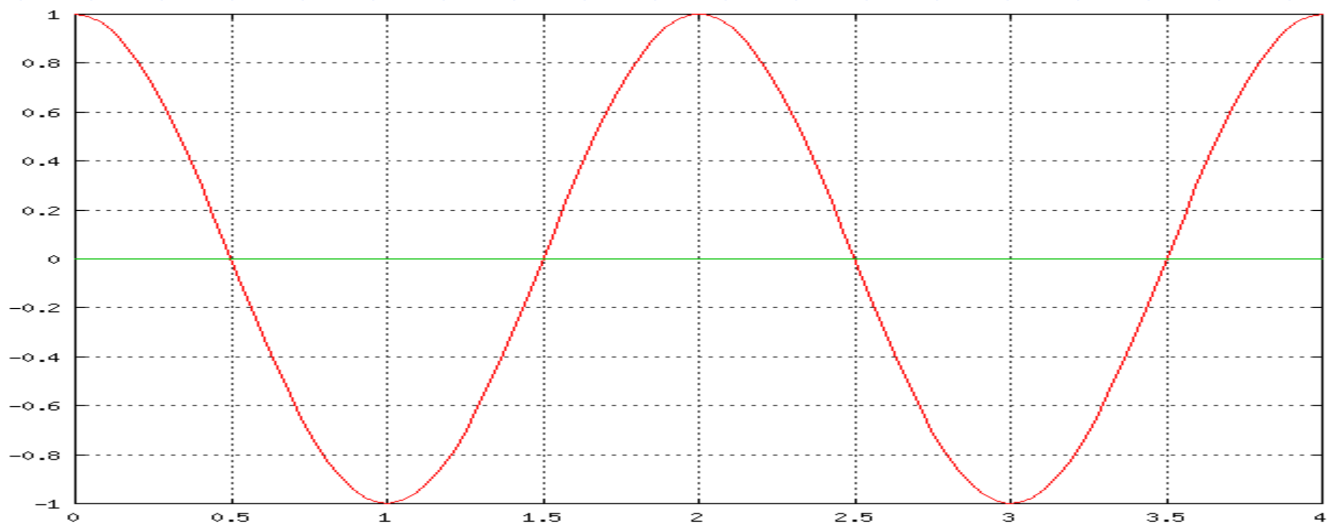
Simple Harmonic Motion

- If we add a mass m to the end of the (massless) spring, stretch it to a displacement x_0, and release it. The spring-mass system will oscillate from x_0 to $-x_0$ and back.



Without friction and air resistance, the oscillation would continue indefinitely

- This is <u>Simple Harmonic Motion</u> (SHM)
- SHM has a maximum magnitude of $|x_0|=A$, called the Amplitude



- One way to understand SHM is to consider the circular motion of a particle and rotational kinematics (The Reference Circle)

- The particle travels on a circle of radius r=A with the line from the center to the particle making an angle θ with respect to the x-axis at some instant in time

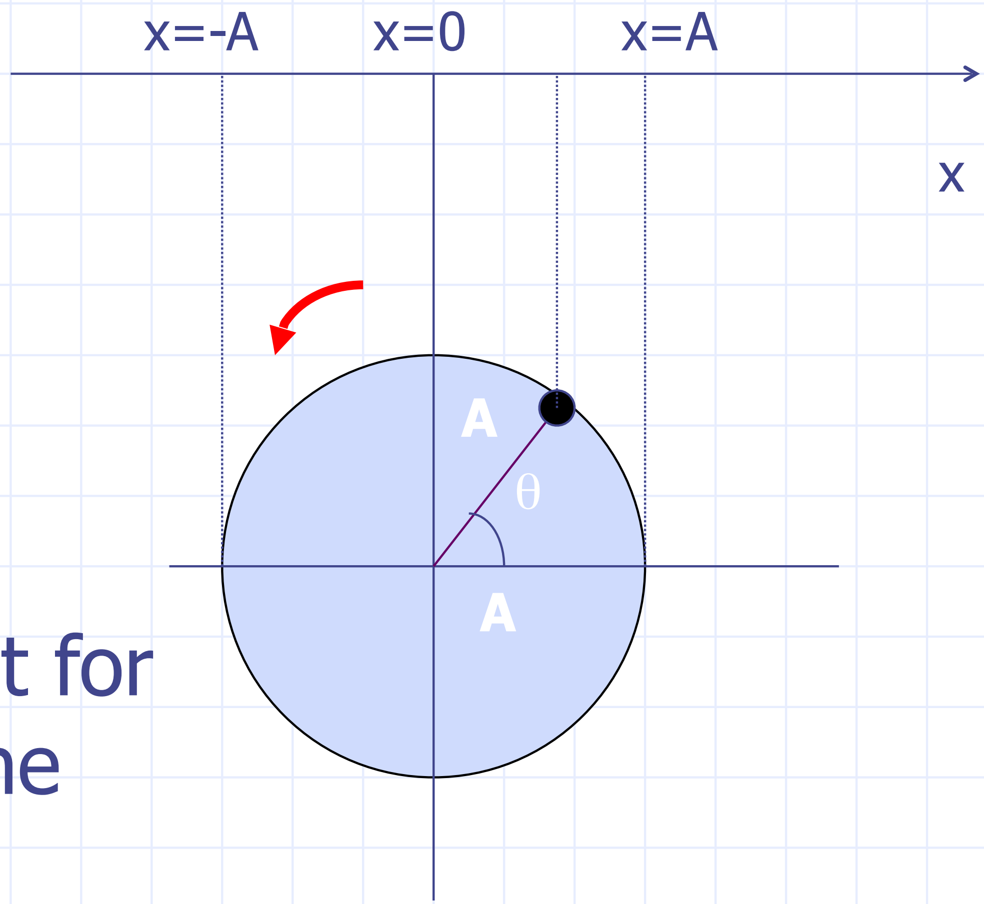
- Now, project this 2D motion onto a 1D axis

$$x = A\cos\theta$$

$$\text{but}\,\theta = \omega t$$

Therefore,

$$\boxed{x = A\cos(\omega t)}$$



- x is the displacement for SHM, which includes the motion of a spring
- SHM is also called sinusoidal motion
- $x_{max}=A=x_0$=amplitude of the motion (maximum)

• ω is the angular frequency (speed) in rad/s. It remains constant during the motion.

- ω and the period T are related

$$\omega = \frac{2\pi}{T}$$

- Define the frequency

$$f = \frac{\#\,\text{cycles}}{\text{sec}} = \frac{1}{T}$$

Units of 1/s = Hertz (Hz)

- $$\omega = \frac{2\pi}{T} = 2\pi\, f$$

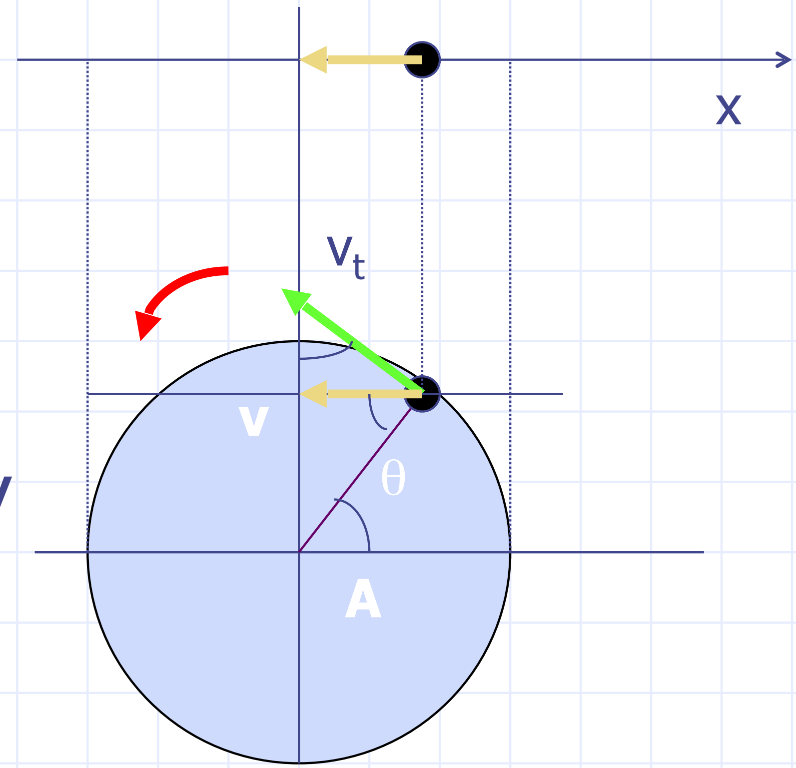
Relates frequency and angular frequency

- As an example, the alternating current (AC) of electricity in the US has a frequency of 60 Hz
- Now, lets consider the velocity for SHM, again using the Reference circle

$$\mathbf{v} = -\mathbf{v}_t \sin\theta$$

$$\mathbf{v}_t = r\omega = A\omega$$

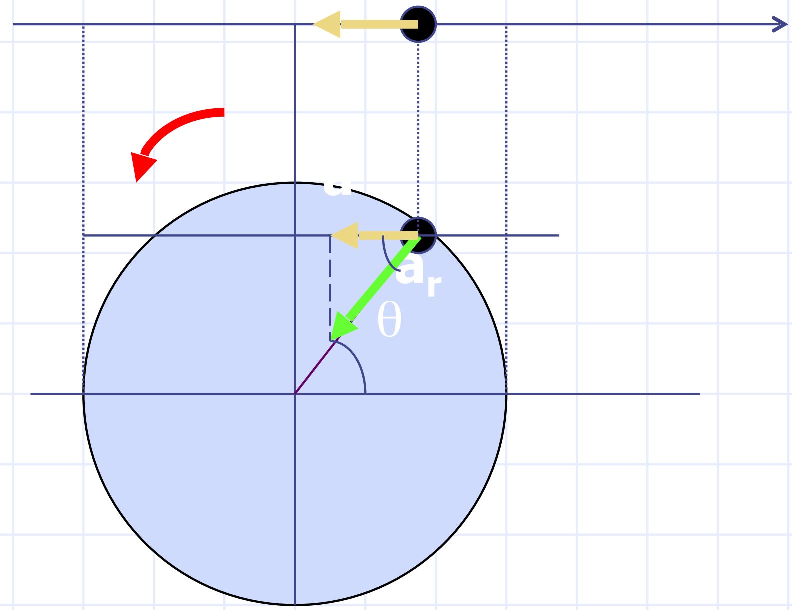
$$\boxed{\mathbf{v} = -\mathbf{A}\omega \sin(\omega t)}$$



- Amplitude of the velocity is

$$\boxed{\mathrm{v}_{\max} = A\omega}$$

- Acceleration of SHM



$$a = -a_r \cos\theta$$

$$a_r = \frac{\mathbf{v}_t^2}{r} = \frac{r^2\omega^2}{r}$$

$$a_r = r\omega^2 = A\omega^2 \Rightarrow \boxed{a = -A\omega^2 \cos(\omega t)}$$

- The amplitude of the acceleration is

$$\boxed{a_{\max} = A\omega^2}$$

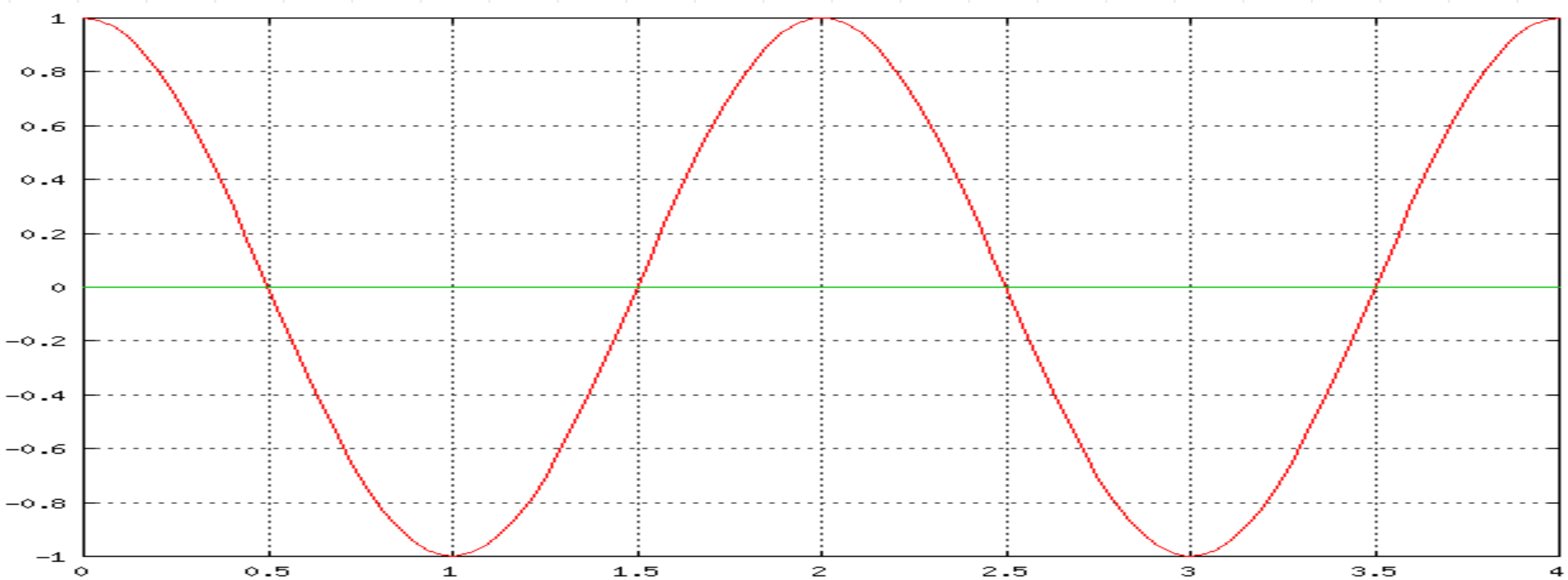
- Summary of SHM

$$x = A\cos(\omega t)$$
$$\mathrm{v} = -A\omega \sin(\omega t)$$
$$a = -A\omega^2 \cos(\omega t)$$

$$x_{\max} = A,\ \omega t = 0, \pi, 2\pi, \ldots$$
$$\mathrm{v}_{\max} = A\omega,\ \omega t = \frac{\pi}{2}, \frac{3\pi}{2} \ldots$$
$$a_{\max} = A\omega^2,\ \omega t = 0, \pi, 2\pi, \ldots$$

x

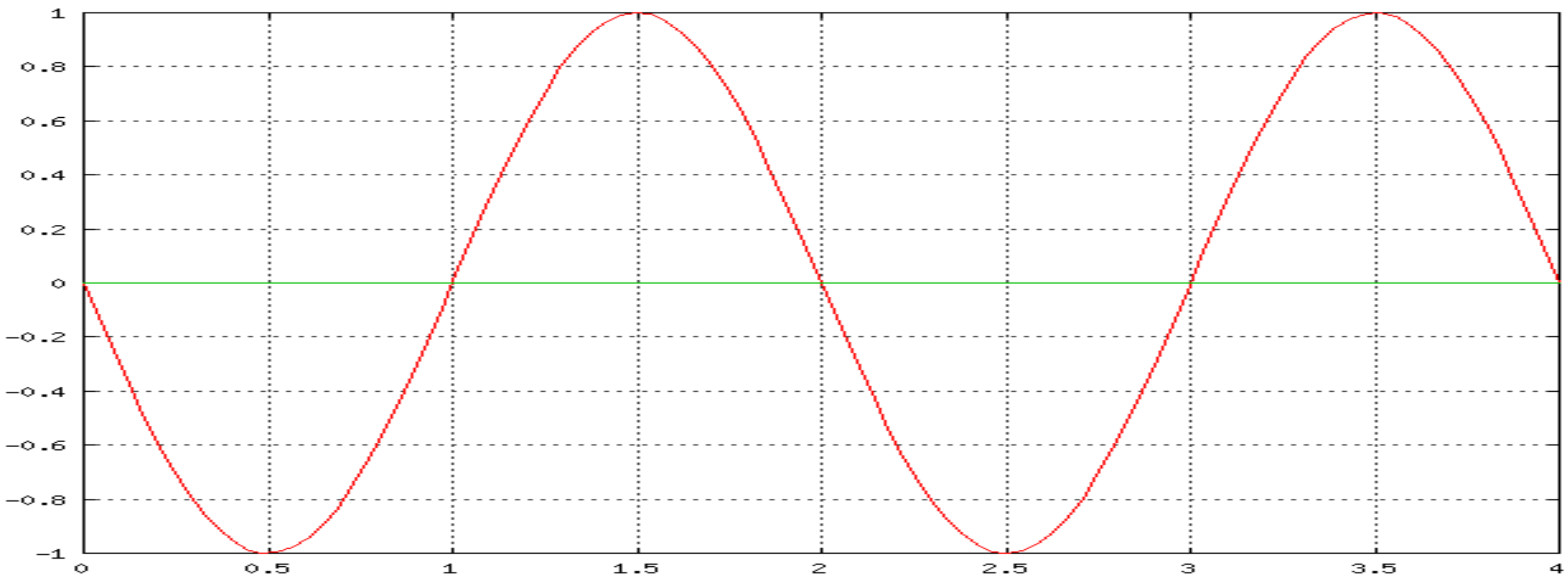
(A=1)



ωt (π rads)

v

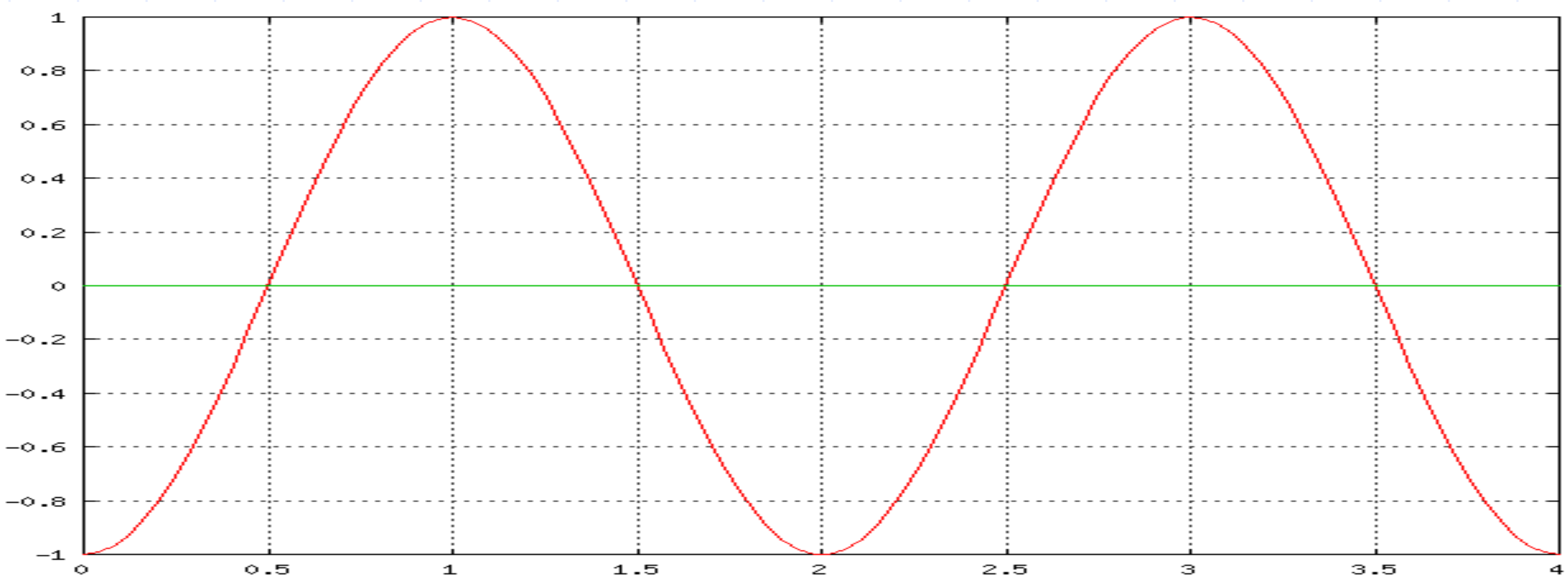
($A\omega=1$)



ωt (π rads)

a

($A\omega^2=1$)



Example Problem

Given an amplitude of 0.500 m and a frequency of 2.00 Hz for an object undergoing simple harmonic motion, determine (a) the displacement, (b) the velocity, and (c) the acceleration at time 0.0500 s.

<u>Solution:</u>

Given: A=0.500 m, f=2.00 Hz, t=0.0500 s.

$$\omega = 2\pi f = 2\pi(2.00\ \mathrm{Hz}) = 4.00\pi\ \mathrm{rad/s}$$

$$\begin{aligned}\omega t &= (4.00\pi\ \mathrm{rad/s})(0.0500\ \mathrm{s}) \\ &= 0.200\pi\ \mathrm{rad} = 0.628\ \mathrm{rad}\end{aligned}$$

(a) $x = A\cos(\omega t) = (0.500\ \mathrm{m})\cos(0.628\ \mathrm{rad})$

$$\boxed{x = 0.405\ \text{m}}$$

$$\text{v} = -A\omega \sin(\omega\, t)$$

$$\text{v} = -(0.500\ \text{m})(4\pi\ \text{rad/s})\sin(0.628\ \text{rad})$$

$$\boxed{\text{v} = -3.69\ \text{m/s}}$$

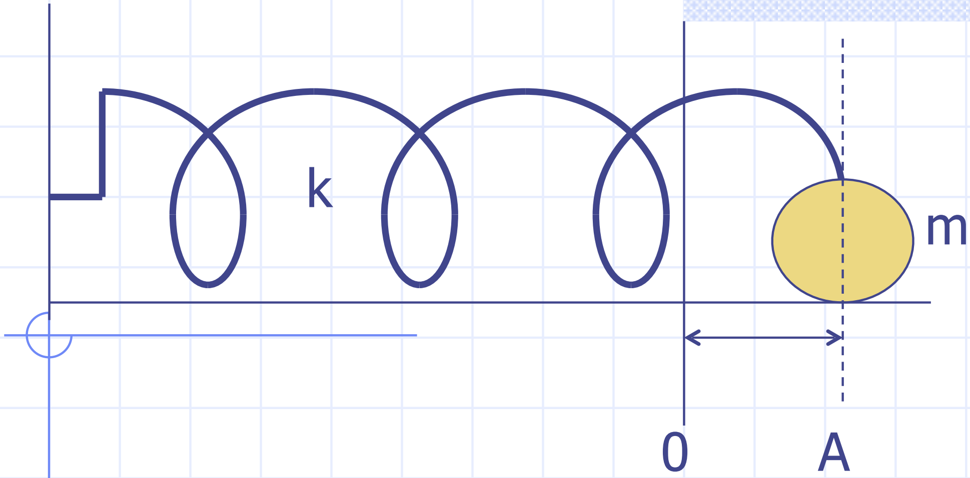
$$a = A\omega^2 \cos(\omega\, t)$$

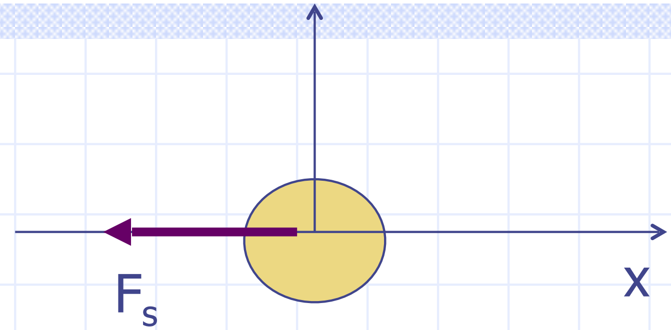
$$a = -(0.500\ \text{m})(4\pi\ \text{rad/s})^2 \cos(0.628\ \text{rad})$$

$$\boxed{a = -63.9\ \text{m/s}^2}$$

Frequency of Vibration

- Apply Newton's 2^{nd} Law to the spring-mass system (neglect friction and air resistance)





$$\sum F_x = ma_x$$

Consider x-direction only

$$F_s = -kx = ma_x$$

Substitute x and a for SHM

$$-k[A\cos(\omega t)] = m[-A\omega^2\cos(\omega t)]$$

$$k = m\omega^2$$

$$\boxed{\omega = \sqrt{\frac{k}{m}}}$$

Angular frequency of vibration for a spring with spring constant k and attached mass m. Spring is assumed to be massless.

$$\omega = 2\pi f \Rightarrow \boxed{f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}}$$

Frequency of vibration

$$T = \frac{1}{f} = \boxed{2\pi\sqrt{\frac{m}{k}} = T}$$

Period of vibration

- This last equation can be used to determine k by measuring T and m

$$T^2 = 4\pi^2\frac{m}{k} \Rightarrow \boxed{k = 4\pi^2\frac{m}{T^2}}$$

- Note that ω (f or T) does not depend on the amplitude of the motion A

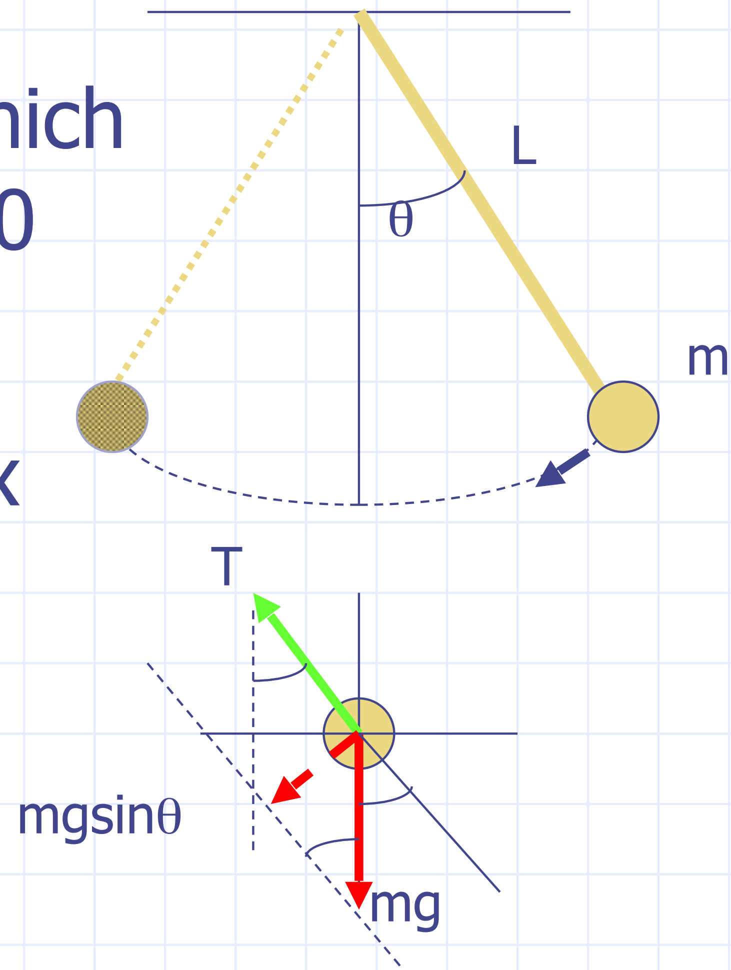
• Can also arrive at these equations by considering derivatives

The Simple Pendulum

- An application of Simple Harmonic Motion
- A mass m at the end of a massless rod of length L
- There is a restoring force which acts to restore the mass to $\theta=0$

$$F = -mg\sin\theta$$

- Compare to the spring $F_s=-kx$
- The pendulum does not display SHM



- But for very small θ (rad), we can make the approximation (θ<0.5 rad or about 25°) → simple pendulum approximation

$$\sin\theta \approx \theta \;\Rightarrow\; F = -mg\theta$$ This is SHM

$$\text{since } s = r\theta = L\theta$$ Arc length

$$F = -mg\frac{s}{L} = -\frac{mg}{L}s$$ Looks like spring force

$$(F_s = -kx) \Rightarrow \boxed{k = \frac{mg}{L}}$$ Like the spring constant

- Now, consider the angular frequency of the spring

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{mg/L}{m}} = \boxed{\sqrt{\frac{g}{L}} = \omega}$$

Simple pendulum angular frequency

$$\boxed{f = \frac{1}{2\pi}\sqrt{\frac{g}{L}}}$$

Simple pendulum frequency

- With this ω, the same equations expressing the displacement x, v, and a for the spring can be used for the simple pendulum, as long as θ is small

- For θ large, the SHM equations (in terms of sin and cos) are no longer valid $\rightarrow$ more complicated functions are needed (which we will not consider)

- A pendulum does not have to be a point-particle