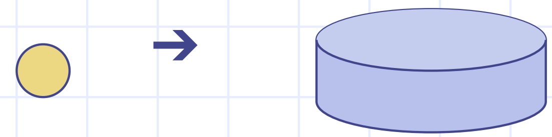
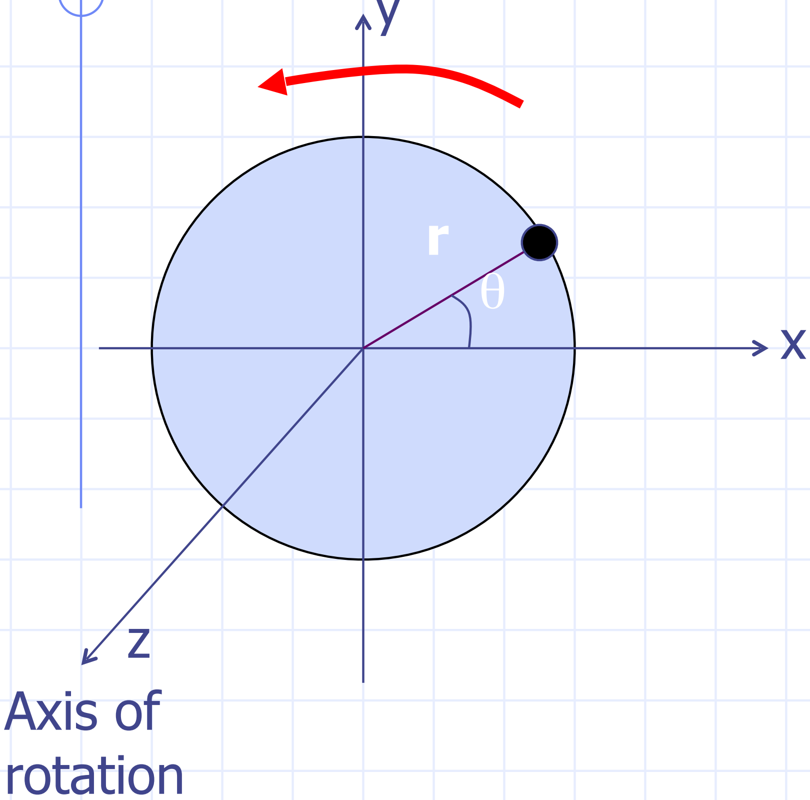
Rotational Kinematics

- Up to now, we have only considered point-particles, i.e. we have not considered their shape or size, only their mass

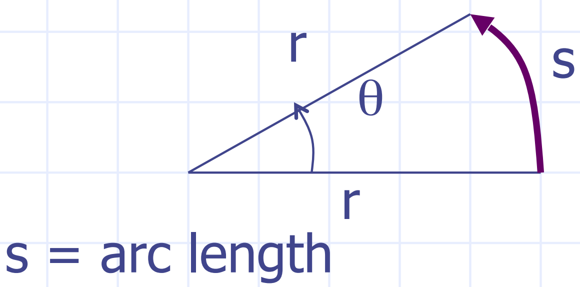


- Also, we have only considered the motion of point-particles – straight-line, free-fall, projectile motion. But real objects can also tumble, twirl, …
- This subject, rotation, is what we explore in this section and in Chapter 9.
- First, we begin by considering the concepts of circular kinematics

- Instead of a point-particle, consider a thin disk of radius r spinning on its axis



- This disk is a real object, it has structure
- We call these kinds of objects Rigid Bodies
- Rigid Bodies do not bend twist, or flex; for example, a billiard ball



$$\theta = \frac{\text{arc length}}{\text{radius}} = \frac{s}{r}$$

Units of radians (rad)

$$s = r\theta$$

- For one complete revolution $\theta = 2\pi$ rad

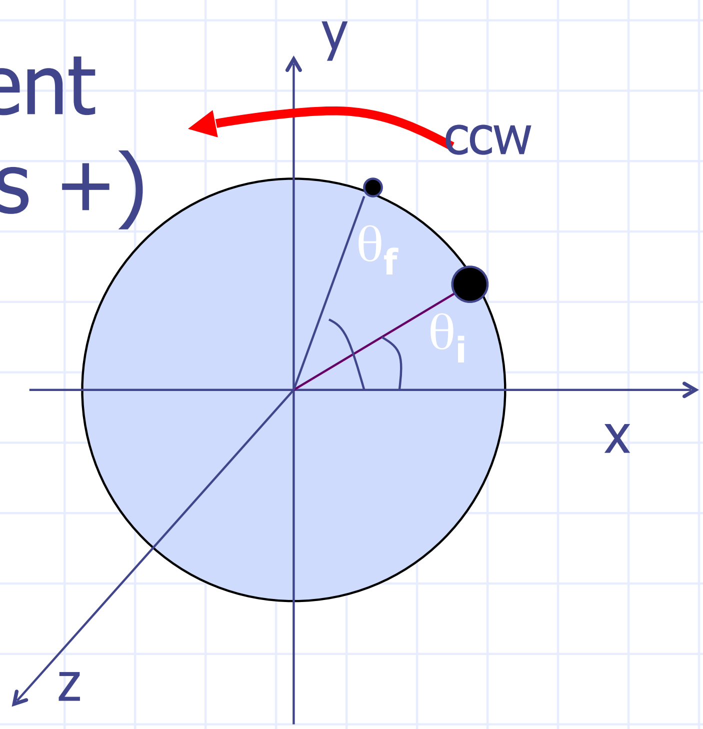
$$s = 2\pi r = \text{circumference}$$

- Conversion relation: 2π rad = 360°

- Now consider the rotation of the disk from some initial angle θ_i to a final angle θ_f during some time period t_i to t_f

$$\Delta\theta = \theta_f - \theta_i$$

Angular displacement (units of rad, ccw is +)



$$\frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t} = \omega_{avg}$$

Average angular velocity (units of rad/s)

- Similar to instantaneous velocity, we can define the Instantaneous Angular Velocity

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

- A change in the Angular Velocity gives

$$\frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta\omega}{\Delta t} = \alpha_{avg}$$

Average Angular Acceleration (rads/s^2)

- Analogous to Instantaneous Angular Velocity, the Instantaneous Angular Acceleration is

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

- Actually, the Angular Velocities and Angular Acceleration are magnitudes of **vector** quantities

$$\vec{\omega} \ \textbf{and} \ \vec{\alpha}$$

- What is their direction?
- They point along the axis of rotation with the sign determined by the right-hand rule

<u>Example</u>

A fan takes 2.00 s to reach its operating angular speed of 10.0 rev/s. What is the average angular acceleration (rad/s^2)?

Solution:

Given: t_f=2.00 s, ω_f=10.0 rev/s

Recognize: t_i=0, ω_i=0, and that ω_f needs to be converted to rad/s

$$\omega_f = 10.0\,\frac{\text{rev}}{\text{s}}\left(\frac{2\pi\text{ rads}}{1\text{ rev}}\right) = 20.0\pi\,\frac{\text{rad}}{\text{s}}$$

$$= 62.8\,\frac{\text{rad}}{\text{s}}$$

$$\alpha_{avg} = \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{20.0\pi - 0}{2.00 - 0} = 10.0\pi\,\frac{\text{rad}}{\text{s}^2}$$

$$\boxed{\alpha_{avg} = 31.4\,\frac{\text{rad}}{\text{s}^2}}$$

Example Problem (you do)

- A centrifuge is a common laboratory instrument that separates components of differing densities in solutions. This is accomplished by spinning a sample around in a circle with a large angular speed. Suppose That after a centrifuge is turned off, it continues to rotate with a constant angular acceleration for 10.2 s before coming to rest. (a) if its initial angular speed was 3850 rpm, what is the magnitude of its angular acceleration? (b) How many revolutions did the centrifuge complete after being turned off?

Equations of Rotational Kinematics

- Just as we have derived a set of equations to describe ``linear” or ``translational” kinematics, we can also obtain an analogous set of equations for rotational motion

- Consider correlation of variables

Translational		Rotational
x	displacement	θ
v	velocity	ω
a	acceleration	α
t	time	t

- Replacing each of the translational variables in the translational kinematic equations by the rotational variables, gives the set of rotational kinematic equations (for constant α)

$$\theta_f = \theta_i + \frac{1}{2}(\omega_i + \omega_f)(t_f - t_i)$$

$$\theta_f = \theta_i + \omega_i(t_f - t_i) + \tfrac{1}{2}\alpha(t_f - t_i)^2$$

$$\omega_f = \omega_i + \alpha(t_f - t_i)$$

$$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)$$

- We can use these equations in the same fashion we applied the translational kinematic equations

Example Problem

A figure skater is spinning with an angular velocity of +15 rad/s. She then comes to a stop over a brief period of time. During this time, her angular displacement is +5.1 rad. Determine (a) her average angular acceleration and (b) the time during which she comes to rest.

Solution:

Given: θ_f=+5.1 rad, ω_i=+15 rad/s

Infer: θ_i=0, ω_f=0, t_i=0

Find: α, t_f ?

(a) Use last kinematic equation

$$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)$$

$$0 = \omega_i^2 + 2\alpha\theta_f$$

$$\alpha = -\frac{\omega_i^2}{2\theta_f} = -\frac{(15\ \text{rad/s})^2}{2(5.1\ \text{rad})} = \boxed{-22\ \frac{\text{rad}}{\text{s}^2}}$$

(b) Use first kinematic equation

$$\theta_f = \theta_i + \tfrac{1}{2}(\omega_i + \omega_f)(t_f - t_i)$$

$$\theta_f = 0 + \tfrac{1}{2}(\omega_i + 0)(t_f - 0)$$

$$t_f = \frac{2\theta_f}{\omega_i} = \frac{2(5.1\ \textbf{rad})}{15\ \textbf{rad/s}} = \boxed{0.68\ \textbf{s}}$$

Or use the third kinematic equation

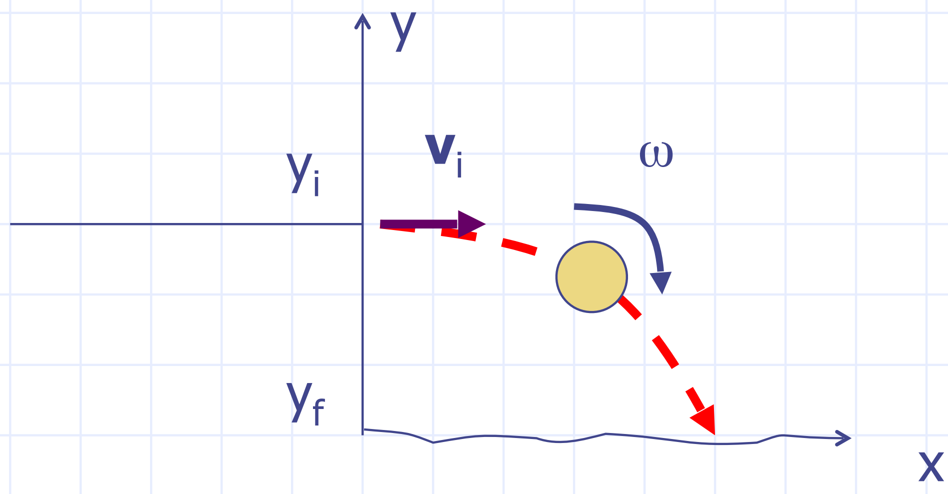
$$\omega_f = \omega_i + \alpha(t_f - t_i)$$

$$0 = \omega_i + \alpha t_f$$

$$t_f = -\frac{\omega_i}{\alpha} = -\frac{15 \text{ rad/s}}{-22 \text{ rad/s}^2} = \boxed{0.68 \text{ s}}$$

Example Problem

At the local swimming hole, a favorite trick is to run horizontally off a cliff that is 8.3 m above the water, tuck into a ``ball,” and rotate on the way down to the water. The average angular speed of rotation is 1.6 rev/s. Ignoring air resistance, determine the number of revolutions while on the way down.



Solution:

Given: $\omega_i = \omega_f = 1.6$ rev/s, $y_i = 8.3$ m

Also, $v_{yi} = 0$, $t_i = 0$, $y_f = 0$

Recognize: two kinds of motion; 2D projectile motion and rotational motion with constant angular velocity.

Method: #revolutions = $\theta = \omega t$. Therefore, need to find the time of the projectile motion, t_f.

Consider y-component of projectile motion since we have no information about the x-component.

$$y_f = y_i + \mathbf{v}_{yi}(t_f\text{-}t_i) + \frac{1}{2}a_y(t_f\text{-}t_i)^2$$

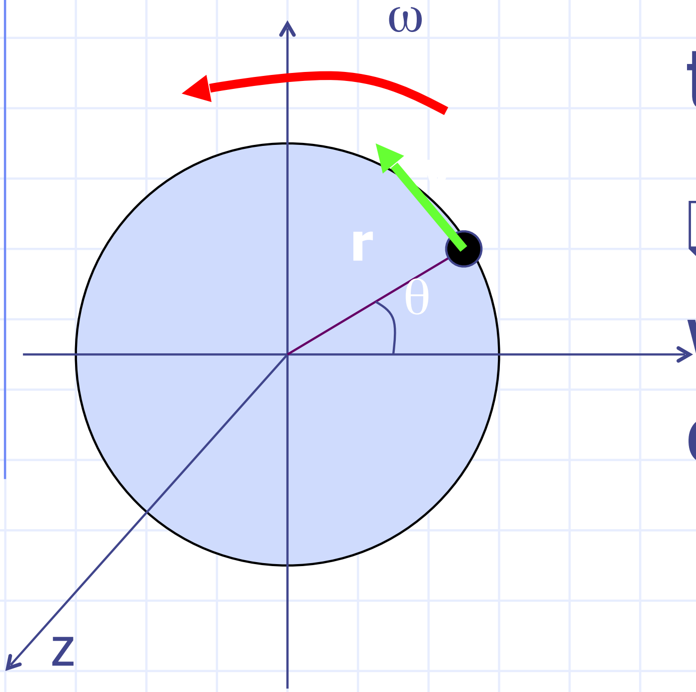
$$0 = y_i - \tfrac{1}{2}gt_f^2 \Rightarrow y_i = \tfrac{1}{2}gt_f^2$$

$$t_f = \sqrt{\frac{2y_i}{g}} = \sqrt{\frac{2(8.3\ \mathbf{m})}{\mathbf{9.80\ \frac{m}{s^2}}}} = 1.3\ \mathbf{s}$$

$$\theta_f = \theta_i + \tfrac{1}{2}(\omega_i + \omega_f)(t_f - t_i) = \omega t_f$$

$$\theta_f = (1.6\ \mathbf{rev/s})(\mathbf{1.3\ s}) = \boxed{\mathbf{2.1\ rev}}$$

Tangential Velocity



- For one complete revolution, the angular displacement is 2π rad
- From Uniform Circular Motion, we know that the time for a complete revolution is a period T
- Therefore the angular velocity (frequency) can be written

$$\omega = \frac{\Delta\theta}{\Delta t} = \boxed{\frac{2\pi}{T} = \omega} \; (\text{rad/s})$$

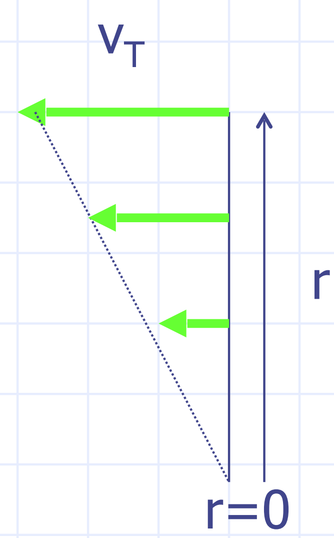
- Also, we know that the speed for an object in a circular path is

$$\mathbf{v} = \frac{2\pi r}{T} = \boxed{r\omega = \mathbf{v}_T}$$

Tangential speed (rad/s)

- The tangential speed corresponds to the speed of a point on a rigid body, a distance r from its center, rotating at an angular speed ω

- Each point on the rigid body rotates at the same angular speed, but its tangential speed depends on its location r



Example Problem

- The Bohr model of the hydrogen atom pictures the electron as a tiny particle moving in a circular orbit about a stationary proton. In the lowest-energy orbit the distance from the proton to the electron is 0.529×10^{-10} m and the tangential (or linear) speed of the electron is 2.18×10^{6} m/s. (a) What is the angular speed of the electron? (b) How many orbits about the proton does it make each second?

- If the angular velocity changes (ω is not constant), then we have an angular acceleration α
  - For some point on a disk, for example

$$\mathbf{v}_{t,i} = r\omega_i, \quad \mathbf{v}_{t,f} = r\omega_f$$

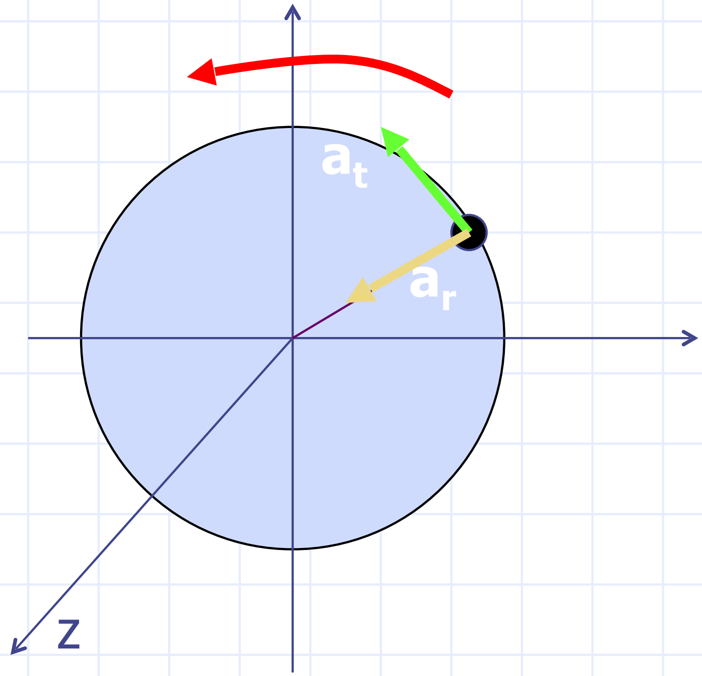
  - From the definition of translational acceleration

$$a = \frac{\mathbf{v}_f - \mathbf{v}_i}{\Delta t} = \frac{r\omega_f - r\omega_i}{\Delta t} = r\left(\frac{\omega_f - \omega_i}{\Delta t}\right)$$

$$\boxed{a_t = r\alpha}$$ Tangential acceleration (units of m/s^2)

- Since the speed changes, this is not Uniform Circular Motion. Also, the Tangential Acceleration is different from the Centripetal Acceleration.

- Recall $a_c = \frac{\mathbf{v}^2}{r} = \frac{\mathbf{v}_t^2}{r} = a_r$

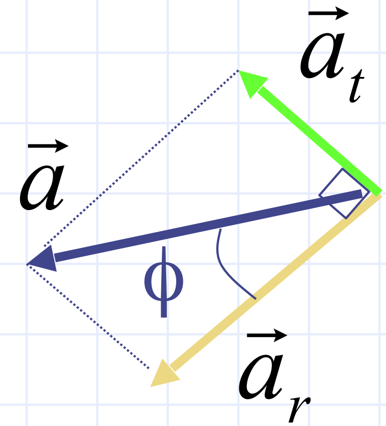


- We can find a total resultant acceleration **a**, since $\mathbf{a}_t$ and $\mathbf{a}_r$ are perpendicular

$$a = \sqrt{a_r^2 + a_t^2}$$

$$\phi = \tan^{-1}(a_t / a_r)$$

- Previously, for the case of uniform circular motion, $a_t=0$ and $a=a_c=a_r$. The acceleration vector pointed to the center of the circle.

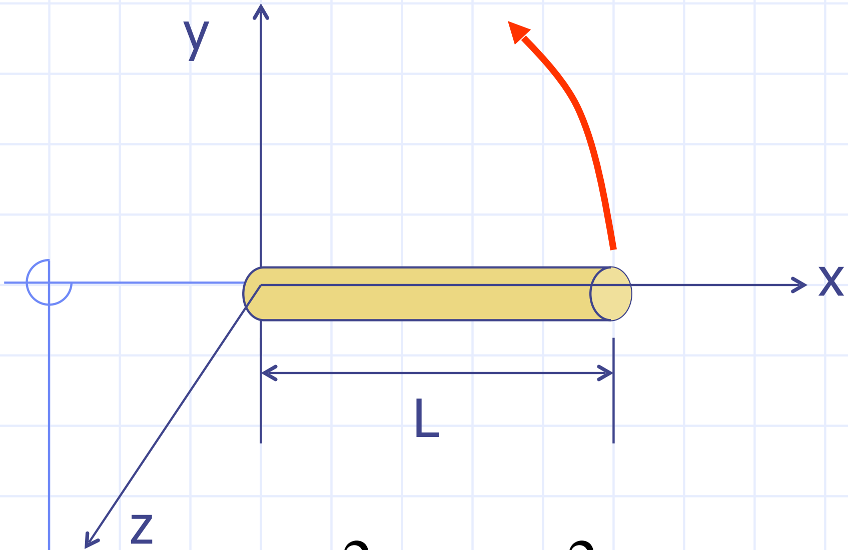


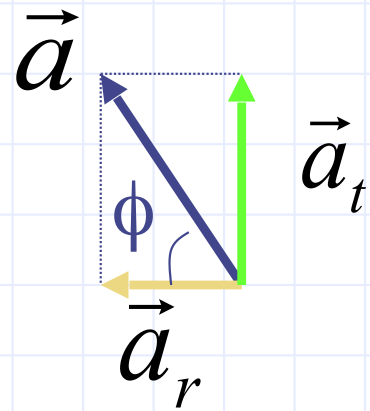
- If $a_t \neq 0$, acceleration points away from the center

Example

A thin rigid rod is rotating with a constant angular acceleration about an axis that passes perpendicularly through one of its ends. At one instant, the total acceleration vector (radial plus tangential) at the other end of the rod makes a 60.0° angle with respect to the rod and has a magnitude of 15.0 m/s^2. The rod has an angular speed of 2.00 rad/s at this instant. What is the rod's length?

Given: $a = 15.0$ m/s^2, $\omega = 2.00$ rad/s (at some time)





$$a_r = \frac{v_t^2}{r} = \frac{v_t^2}{L}, \qquad a_t = r\alpha = L\alpha$$

$$v_t = r\omega = L\omega \quad \Rightarrow a_r = \frac{(L\omega)^2}{L} = L\omega^2$$

$$a_r = a\cos\phi = L\omega^2$$

Solve for L

$$L = \frac{a\cos\phi}{\omega^2} = \frac{(15.0\ \frac{\text{m}}{\text{s}^2})\cos 60.0^\circ}{(2.00\,\frac{\text{rad}}{\text{s}})^2} = \boxed{1.88\ \text{m}}$$