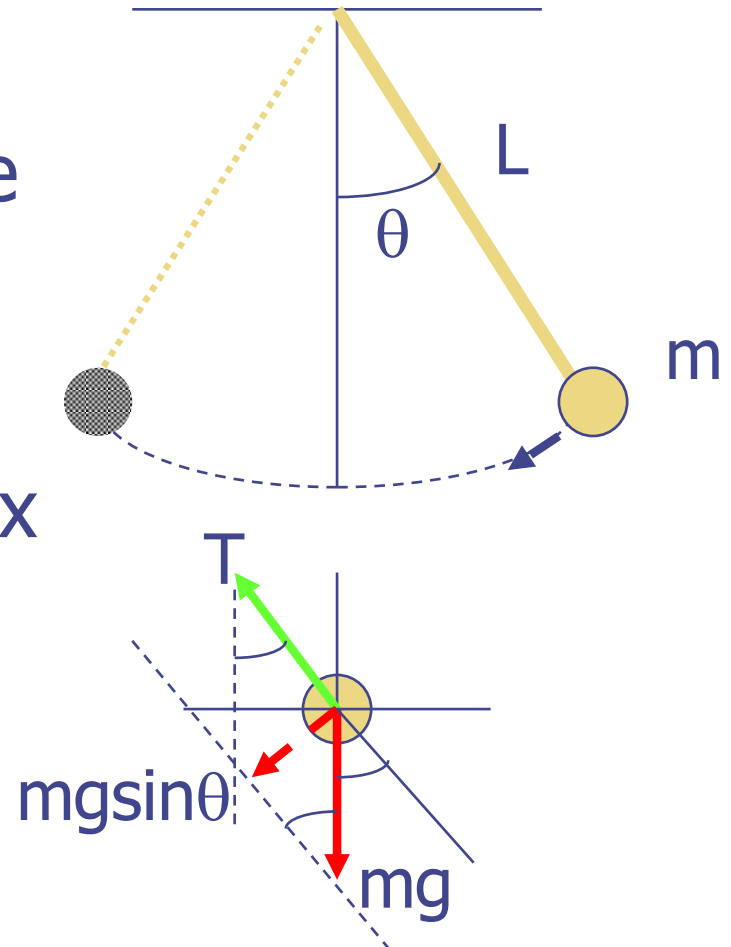# The Simple Pendulum

❑ An application of Simple Harmonic Motion

❑ A mass $m$ at the end of a massless rod of length $L$

❑ There is a <u>restoring force</u> which acts to restore the mass to $\theta=0$

$$F = -mg\sin\theta$$

• Compare to the spring $F_s=-kx$

• The pendulum *does not* display SHM

❑ But for very small $\theta$ (rad), we can make the approximation ($\theta < 0.5$ rad or about 25° )

$\to$ *simple pendulum* approximation

$\sin\theta \approx \theta \Rightarrow F = -mg\theta$  This is SHM

since $s = r\theta = L\theta$  Arc length

$F = -mg\dfrac{s}{L} = -\dfrac{mg}{L}s$  Looks like spring force

$(F_s = -kx) \Rightarrow \boxed{k = \dfrac{mg}{L}}$  Like the spring constant

❑ Now, consider the angular frequency of the spring

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{mg/L}{m}} = \boxed{\sqrt{\frac{g}{L}} = \omega}$$

Simple pendulum angular frequency

$$\boxed{f = \frac{1}{2\pi}\sqrt{\frac{g}{L}}}$$

Simple pendulum frequency

• With this $\omega$, the same equations expressing the displacement $x$, $v$, and $a$ for the spring can be used for the simple pendulum, as long as $\theta$ is small

• For $\theta$ large, the SHM equations (in terms of sin and cos) are no longer valid → more complicated functions are needed (which we will not consider)

• A pendulum does not have to be a point-particle

# The Physical Pendulum

❑ A rigid body can also be a pendulum

❑ The simple pendulum has a moment of inertia
$$I = mL^2$$

❑ Rewrite $\omega$ in terms of $I$

$$\omega = \sqrt{\frac{g}{L}} = \sqrt{\frac{mg}{mL}} = \sqrt{\frac{mgL}{mL^2}}$$

$$\boxed{\omega = \sqrt{\frac{mgL}{I}}} \quad \text{or} \quad \boxed{f = \frac{1}{2\pi}\sqrt{\frac{mgL}{I}}}$$

L

cg

m

mg
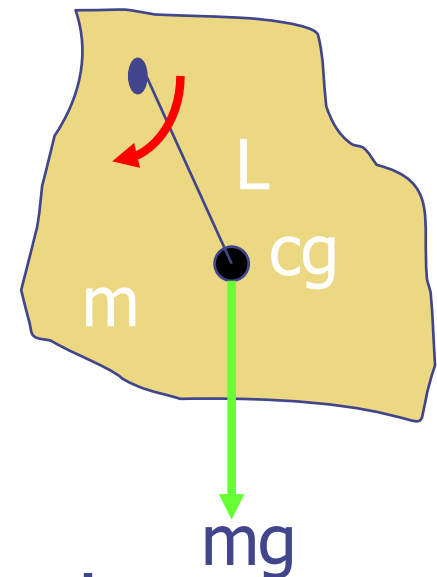
❑ $L$ is the distance from the rotation axis
   to the center of gravity

# Example

Use a thin disk for a simple physical pendulum with rotation axis at the rim. a) find its period of oscillation and b) the length of an equivalent simple pendulum.
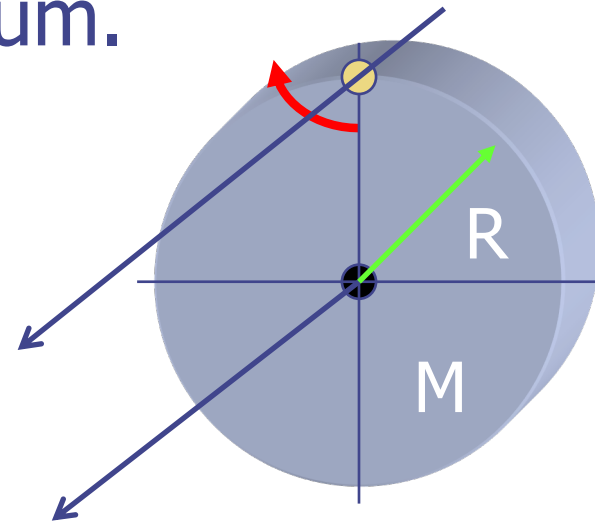
Solution:

a) From table 12.2

$$I_{cm} = \tfrac{1}{2}MR^2$$

R

M

But we need $I$ at the rim, so
   apply parallel axis theorem, $h=R$

$$I_{rim} = I_{cm} + MD^2 = \tfrac{1}{2}MR^2 + MR^2 = \tfrac{3}{2}MR^2$$

Since physical pendulum frequency is

$$f = \frac{1}{2\pi}\sqrt{\frac{MgL}{I}} \Rightarrow T = 2\pi\sqrt{\frac{I}{MgL}}$$

Distance from rotation axis to cm: L=R

$$T = 2\pi\sqrt{\frac{\frac{3}{2}MR^2}{MgR}} = 2\pi\sqrt{\frac{3R}{2g}}$$

Let R=0.165 m (6.5 inches)

$$T = 2\pi\sqrt{\frac{3(0.165)}{2(9.8)}} = 0.999s$$

Would make a good clock!

Note that the period or frequency of a pendulum does not depend on the mass and would be different on other planets

b) For an equivalent simple pendulum, we need the simple and disk pendulums to have the same period

$$T_{disk} = 2\pi\sqrt{\frac{3R}{2g}} = T_{sp} = 2\pi\sqrt{\frac{L}{g}}$$

$$\sqrt{\frac{3R}{2g}} = \sqrt{\frac{L}{g}} \Rightarrow \frac{3R}{2g} = \frac{L}{g}$$

$$\boxed{L = \frac{3R}{2}} = \frac{3(0.165m)}{2} = \boxed{0.248m}$$

# Damped Harmonic Motion

❑ Simple harmonic motion in which the amplitude is steadily decreased due to the action of some non-conservative force(s), i.e. friction or air resistance (**F**=-b**v**, where b is the damping coefficient)

❑ 3 classifications of damped harmonic motion:

1. *Underdamped* – oscillation, but amplitude decreases with each cycle (shocks)
2. *Critically damped* – no oscillation, with smallest amount of damping
3. *Overdamped* – no oscillation, but more damping than needed for critical

- Apply Newton's 2nd Law

$$\sum F_x = -kx - bv = ma_x$$
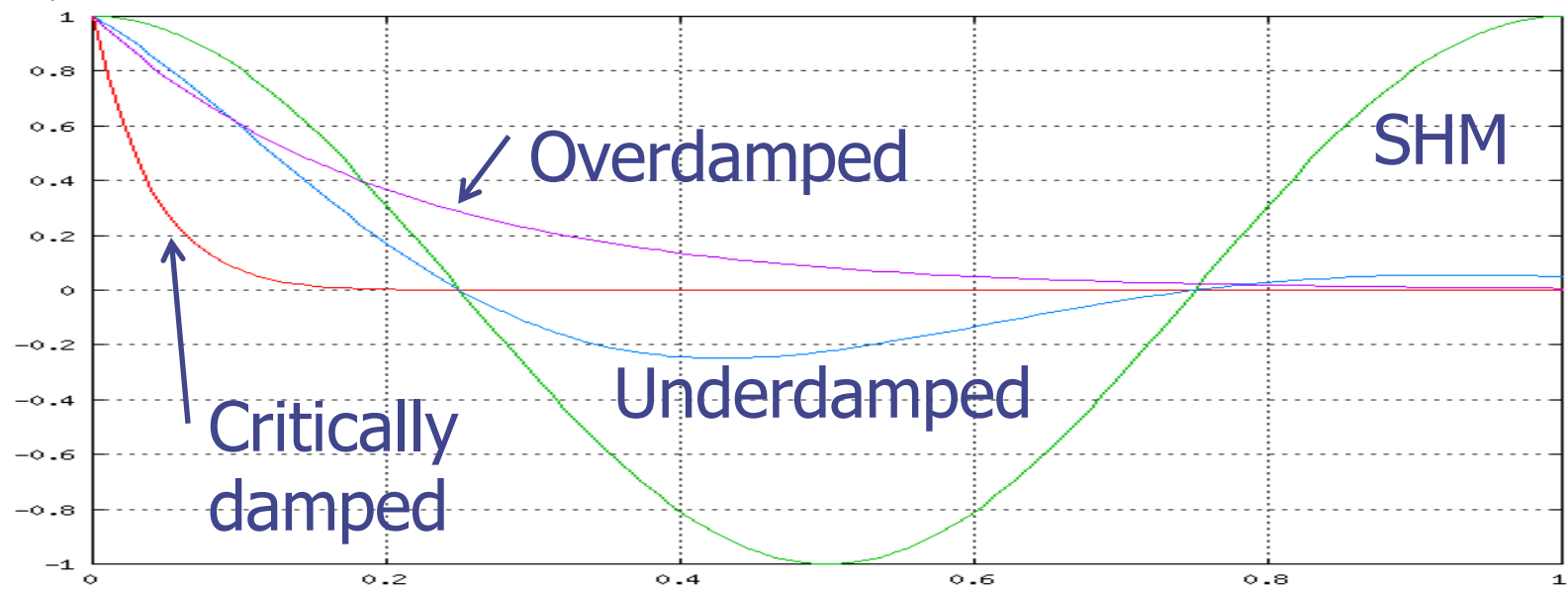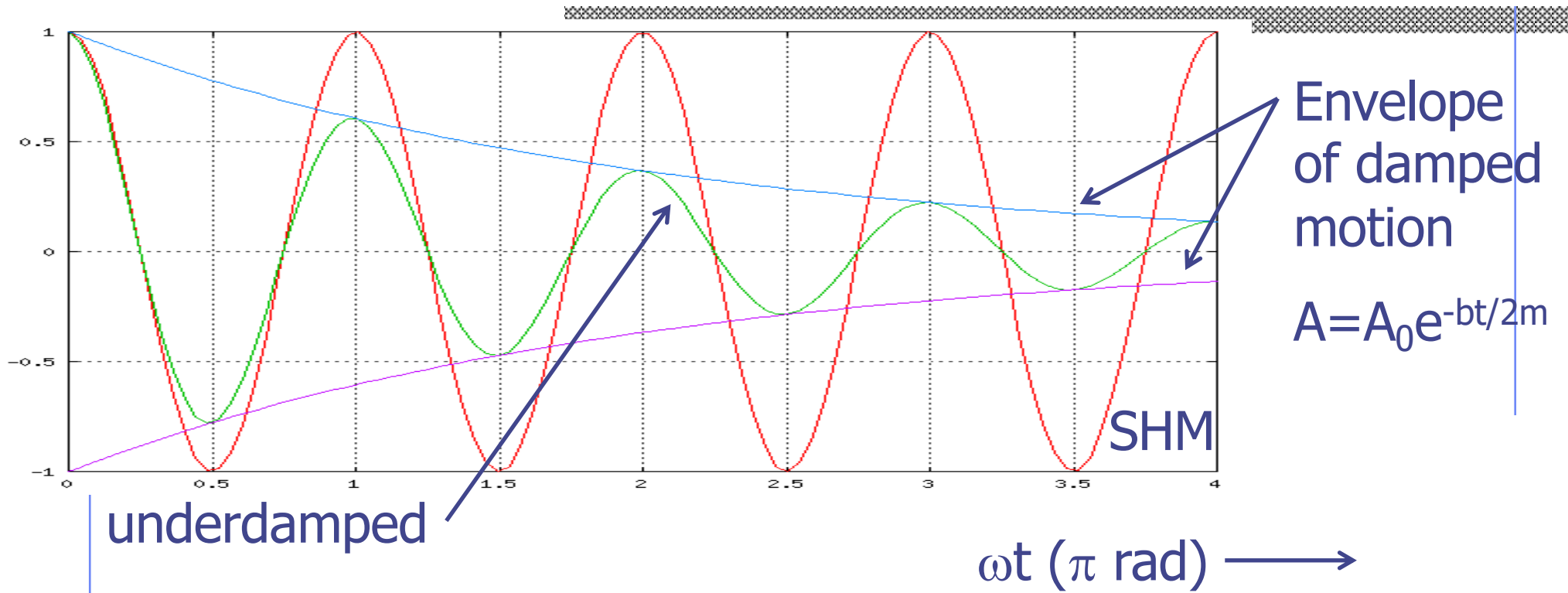
$$-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

- Another 2nd-order ordinary differential equation, but with a 1st-order term

- The solution is $x = A_0 e^{-\frac{b}{2m}t}\cos(\omega t + \phi_0)$

- Where $\omega = \sqrt{\omega_0^2 - (b/2m)^2}$  $\omega_0 = \sqrt{k/m}$

- Type of damping determined by comparing $\omega_0$ and $b/2m$

Envelope of damped motion

$$A = A_0 e^{-bt/2m}$$

SHM

underdamped

$\omega t \ (\pi \ \text{rad}) \longrightarrow$

Overdamped

SHM

Underdamped

Critically damped

# Forced Harmonic Motion

❑ Unlike damped harmonic motion, the amplitude may increase with time

❑ Consider a swing (or a pendulum) and apply a force that increases with time; the amplitude will increase with time

Forced HM

SHM

$\omega t \ (\pi \ \text{rad}) \longrightarrow$
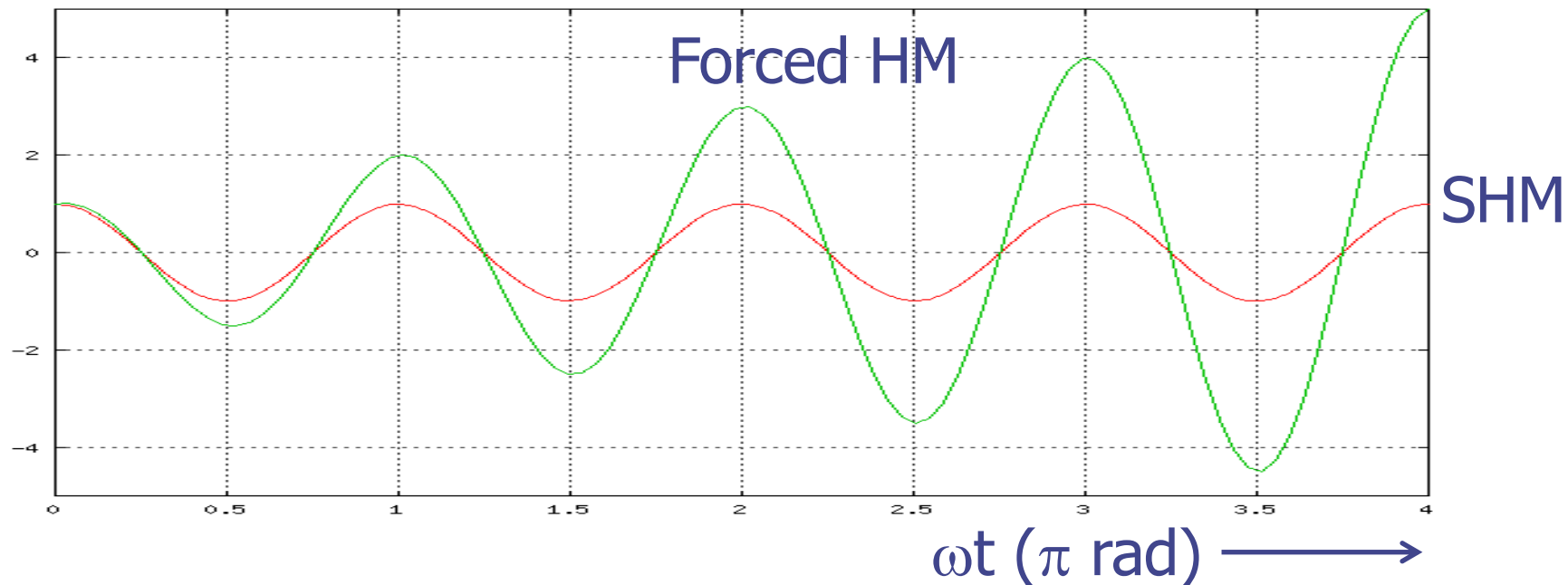
❑ Consider the spring-mass system, it has a frequency

$$f = \frac{1}{2\pi}\sqrt{k/m} = f_0$$

❑ We call this the *natural frequency $f_0$* of the system. All systems (car, bridge, pendulum, etc.) have an $f_0$

❑ We can apply a time-dependent external driving force with frequency $f_d$ ($f_d \neq f_0$) to the spring-mass system

$$F(t) = F_{ext}\cos(2\pi \ f_d t)$$

❑ This is forced harmonic motion, the amplitude increases

❑ But if $f_d = f_0$, the amplitude can increase dramatically – this is a condition called *resonance*

❑ Examples: a) out-of-balance tire shakes violently at certain speeds,

b) Tacoma-Narrows bridge's $f_0$ matches frequency of wind