Chap. 11: Momentum

- As shown already, velocity is a useful kinematic property
- A property that is more useful (as we will see) is the linear momentum defined as

$$\vec{p} \approx m\vec{v}$$

- The momentum vector clearly points in the same direction as the velocity vector
- The units for momentum are kg m/s. There is no derived unit
- However, this relation is approximate and only valid when

$$|\vec{v}| << c$$

Example

A car of mass 750 kg is traveling east at a speed of 10.0 m/s. The car hits a wall and rebounds (moving west) with a speed of 0.100 m/s. Determine its momentum before and after the impact. Determine the impulse (later).

Solution:

Given: m = 750 kg,

$$\vec{\mathbf{v}}_i = \mathbf{10.0}\,\tfrac{\mathbf{m}}{\mathbf{s}}\,\hat{i}$$

$$\vec{\mathbf{v}}_f = 0.10\,\tfrac{\mathbf{m}}{\mathbf{s}}\ \mathbf{west} = \mathbf{-0.10}\,\tfrac{\mathbf{m}}{\mathbf{s}}\,\hat{i}$$

$$\vec{p}_i = m\vec{v}_i = (750 \text{ kg})(10.0 \text{ m/s } \hat{i})$$
$$= 7.50\text{x}10^3 \text{ kg m/s } \hat{i}$$

$$\vec{p}_f = m\vec{v}_f = (750 \text{ kg})(-0.100 \text{ m/s } \hat{i})$$
$$= -7.50\text{x}10^1 \text{ kg m/s } \hat{i}$$

- Change in momentum vector is

$$\Delta\vec{p} = \vec{p}_f - \vec{p}_i$$
$$= (-75\hat{i}) - 7500\hat{i} = -7575\hat{i} \quad \text{kg m/s}$$

Impulse and Momentum

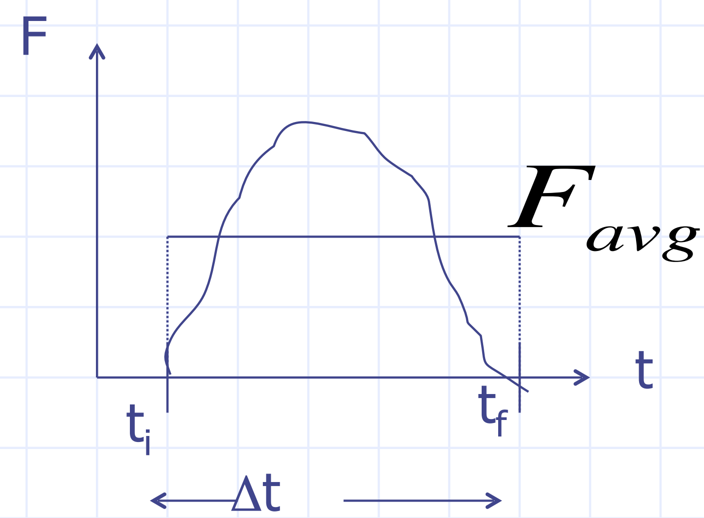
- Lets consider a force which has a time duration (usually short) and with a magnitude that may vary with time – examples: a bat hitting a baseball, a car crash, a asteroid or comet striking the Earth, etc.

- It is difficult to deal with a time-varying force, so we might take the mean value

- Define a new quantity, the impulse

$$\vec{F}_{avg}\Delta t = \vec{J}$$

- a vector, points in the same direction as the force
- has units of N s



$$\sum \vec{F} = m\vec{a} = m\frac{d\vec{\mathbf{v}}}{dt} = \frac{d(m\vec{\mathbf{v}})}{dt}$$

$$\boxed{\sum \vec{F} = \frac{d\vec{p}}{dt}}$$

- Assume only one force acts (or net force), then

$$d\vec{p} = \vec{F}dt \Rightarrow \int_{\vec{p}_i}^{\vec{p}_f} d\vec{p} = \int_{t_i}^{t_f} \vec{F}dt$$

$$\vec{p}_f - \vec{p}_i = \int_{t_i}^{t_f} \vec{F}dt \quad \text{or} \quad \boxed{\vec{J} = \int_{t_i}^{t_f} \vec{F}dt = \Delta\vec{p}}$$

- The Impulse-momentum Theorem

- Alternatively, consider the definition of acceleration

$$\vec{a} = \frac{\vec{\mathrm{v}}_f - \vec{\mathrm{v}}_\mathrm{i}}{\Delta \mathrm{t}} \Rightarrow m\vec{a} = \frac{m\vec{\mathrm{v}}_f - m\vec{\mathrm{v}}_\mathrm{i}}{\Delta \mathrm{t}}$$

$$\vec{F}\Delta t = \vec{p}_f - \vec{p}_i \quad \text{or} \quad \boxed{\vec{J} = \Delta\vec{p}}$$

- The Impulse-Momentum Theorem says that if an impulse (force*time duration) is applied to an object, its momentum changes

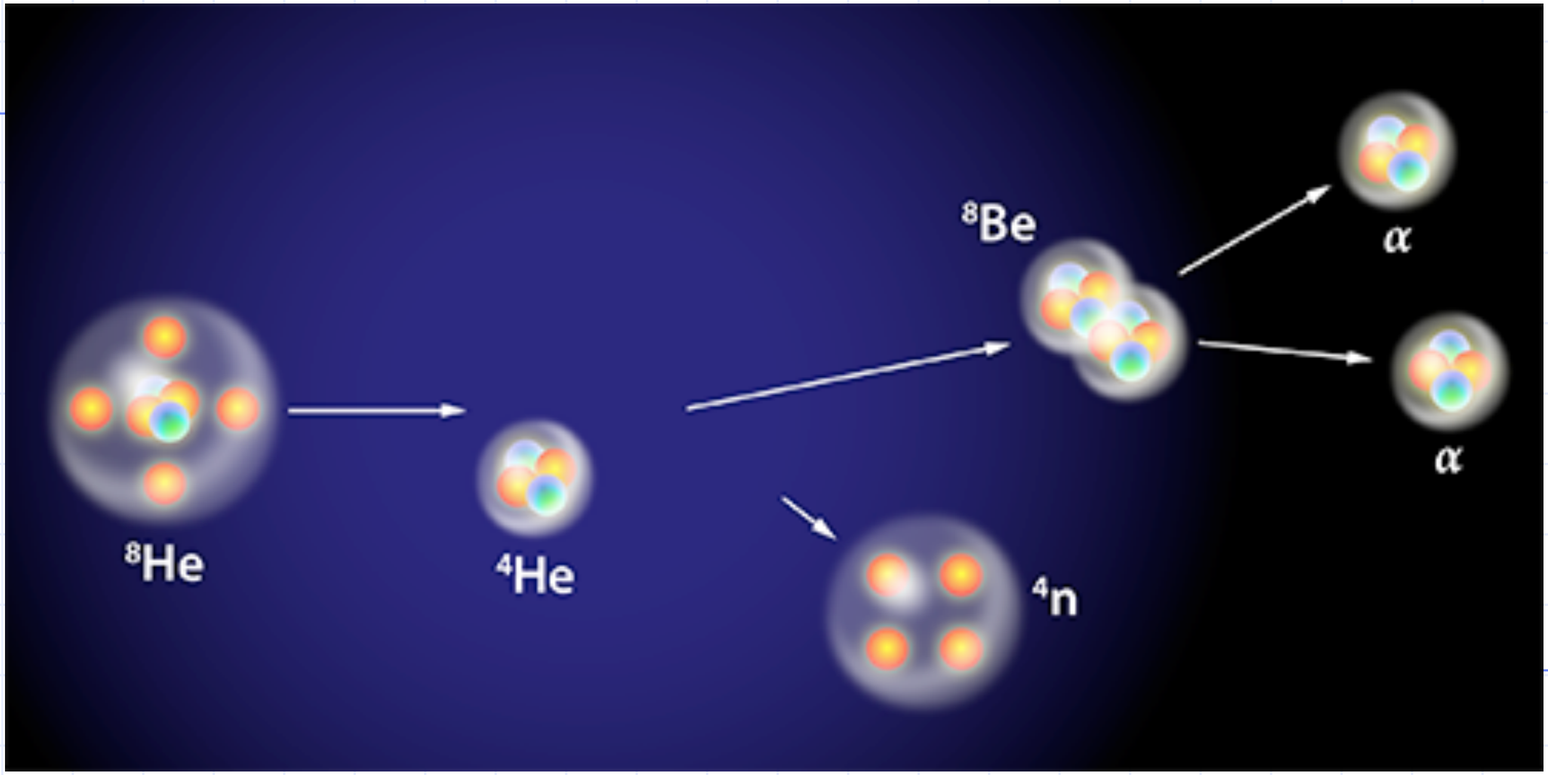
- In this example, the impact of the car with the wall applies an impulse to the car → car's p changes

$$\vec{J} = \vec{p}_f - \vec{p}_i = (-75.0\hat{i}) - 7.50\mathrm{x}10^3\hat{i}$$

$$= \boxed{-7.58\mathrm{x}10^3 \, \frac{\mathrm{kg\ m}}{\mathrm{s}} \, \hat{i}}$$

Tetraneutron? (aside)

- Can a four-neutron, no-proton particle exist?



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- A 186 MeV/u beam of radioactive ^8He (doubly-charged) collides with normal ^4He
- Experiments done at RIKEN Radioactive Ion Beam Factory
- Isolated neutrons have a half-life of only about 15 minutes

$$n^0 \rightarrow p^+ + e^- + \bar{\nu}_e$$

- Proton has a half-life of 10^{29} years
- Bound neutron in a nucleus are stable
- Conserved properties of matter: charge, mass, quark number (proton number, neutron number), lepton number, …

Momentum Conservation and Collisions

- Involves two (or more) objects which may have their motion (velocity, momentum) altered by collisions
- These concepts are applicable to the collisions of atoms, ions, billiard balls, cars, planetary objects, galaxies, etc.
- Say, we have a collection of interacting particles numbered 1, 2, 3, … We can define the Total Momentum of the system (all the particles) as just the sum of all the individual momenta

$$\vec{P} = \sum \vec{p}_i = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \ldots$$

- Imagine that these particles interact in some way – collide and scatter

- As long as there are no net external forces acting on the system (collection of objects), the Total Linear Momentum does not change

- This means: the Total Linear Momentum is the same before the collision, during the collision, and after the collision

$$\vec{P}_i = \vec{P}_f$$

- **Conservation of Linear Momentum**

Example - 1D

A 45-kg swimmer runs with a horizontal velocity of +5.1 m/s off of a boat dock into a stationary 12-kg rubber raft. Find the velocity that the swimmer and raft would have after impact, if there were no friction and resistance due to the water.

Solution:

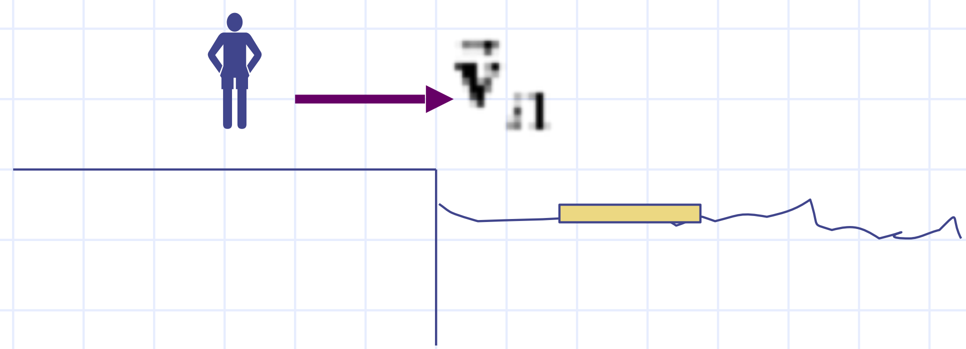
Given: $m_1 = 45$ kg, $m_2 = 12$ kg,

$$\vec{\mathrm{v}}_{i1} = +5.1\frac{\mathrm{m}}{\mathrm{s}}\hat{i}, \vec{\mathrm{v}}_{i2} = 0$$

Find: $\vec{v}_{f1}, \vec{v}_{f2} = ?$

- Consider motion of boy and raft just before and just after impact
- Boy and raft define <u>the system</u>
- Neglect friction and air resistance → no net external forces

• Therefore, we can use the Conservation of Linear Momentum



$$\vec{P}_i = \vec{P}_f$$

$$\vec{p}_{i1} + \vec{p}_{i2} = \vec{p}_{f1} + \vec{p}_{f2}$$

$$m_1\vec{\mathbf{v}}_{i1} + m_2\vec{\mathbf{v}}_{i2} = m_1\vec{\mathbf{v}}_{f1} + m_2\vec{\mathbf{v}}_{f2}$$

- Since, the boy moves with the raft after the impact

$$\vec{v}_{f1} = \vec{v}_{f2} = \vec{v}_f$$

$$m_1 \mathrm{v}_{i1} = (m_1 + m_2)\mathrm{v}_f$$

$$\mathrm{v}_f = \frac{m_1}{(m_1 + m_2)}\mathrm{v}_{i1} = \frac{45}{(45+12)}5.1$$

$$= 4.0\,\frac{\mathrm{m}}{\mathrm{s}}$$

$$\boxed{\vec{\mathrm{v}}_f = +4.0\,\frac{\mathrm{m}}{\mathrm{s}}\,\hat{i}} = \vec{\mathrm{v}}_{f1} = \vec{\mathrm{v}}_{f2}$$

- What if we have the case where $v_{f1} \neq v_{f2}$? We then have two unknowns. So, we need another equation. We will return to this prob. in Chap. 10.

Example Problem

- Suppose that all the people of the Earth go to the North Pole and, on a signal, all jump straight up. Estimate the recoil speed of the Earth. The mass of the Earth is 6×10^{24} kg.

Example Problem

- When they are far apart, the momentum of a proton is $<3.4,0,0> \times 10^{-21}$ kg m/s as it approaches another proton that is initially at rest. The two protons repel each other electrically, without coming close enough to touch. When they are once again far apart, one of the protons now has a momentum $<2.4,1.55,0> \times 10^{-21}$ kg m/s. At that instance, what is the momentum of the other proton?

Collisions in 2D

- Start with Conservation of Linear Momentum vector equation
- Similar to 2D kinematic problems: break into x- and y-components

$$\vec{P}_i = \vec{P}_f$$

$$P_{ix} = P_{fx}, \qquad P_{iy} = P_{fy}$$

<u>Example Problem</u>

Three guns are aimed at the center of a circle. They are mounted on the circle, 120° apart. They fire in a timed sequence, such that the three bullets collide at the center and mash into a stationary lump.

Two of the bullets have identical masses of 4.50 g each and speeds of v_1 and v_2. The third bullet has a mass of 2.50 g and a speed of 575 m/s. Find the unknown speeds.

Solution:

Given: $m_1 = m_2 = 4.50$ g, $m_3 = 2.50$ g,
$v_{i3} = 575$ m/s, $v_{f1} = v_{f2} = v_{f3} = 0$

Find: v_{i1} and v_{i2}

Method: If we neglect air resistance $\rightarrow$ then there are no external forces (in the horizontal x-y plane; gravity acts in the vertical direction) $\rightarrow$ we can use Conservation of Linear Momentum

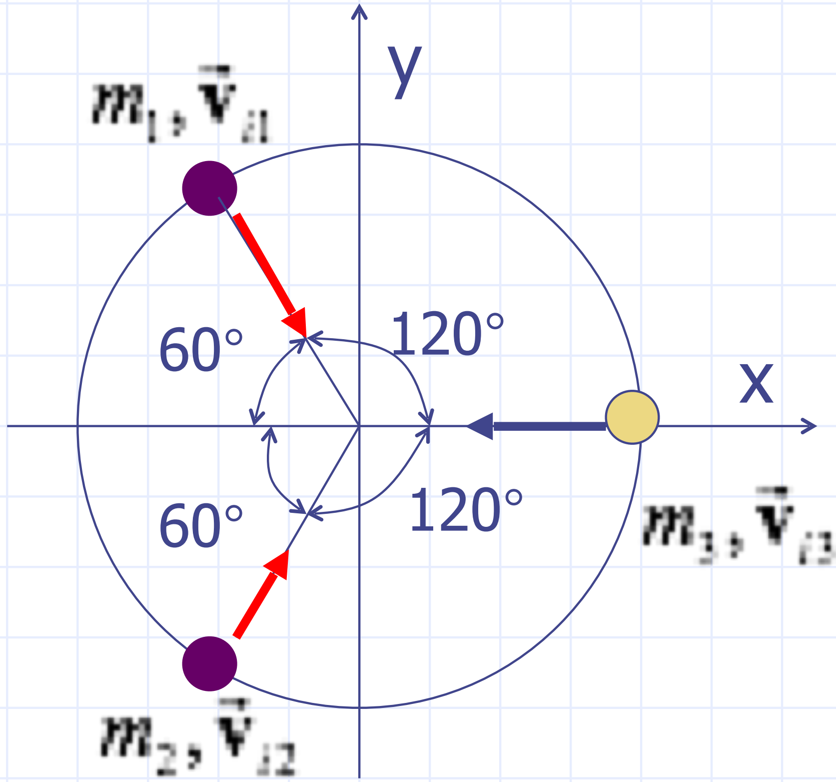
$$\vec{P}_i = \vec{P}_f$$

$$P_{iy} = P_{fy}$$

$$-m_1 \mathbf{v}_{i1} \sin 60^\circ + m_2 \mathbf{v}_{i2} \sin 60^\circ = 0$$

$$\boxed{\mathbf{v}_{i1} = \mathbf{v}_{i2}}$$

$$P_{ix} = P_{fx}$$



$$m_1 \mathbf{v}_{i1} \cos 60^\circ + m_2 \mathbf{v}_{i2} \cos 60^\circ - m_3 \mathbf{v}_{i3} = 0$$

$$m_1 \mathbf{v}_{i1}\left(\tfrac{1}{2}\right) + m_1 \mathbf{v}_{i1}\left(\tfrac{1}{2}\right) = m_3 \mathbf{v}_{i3}$$

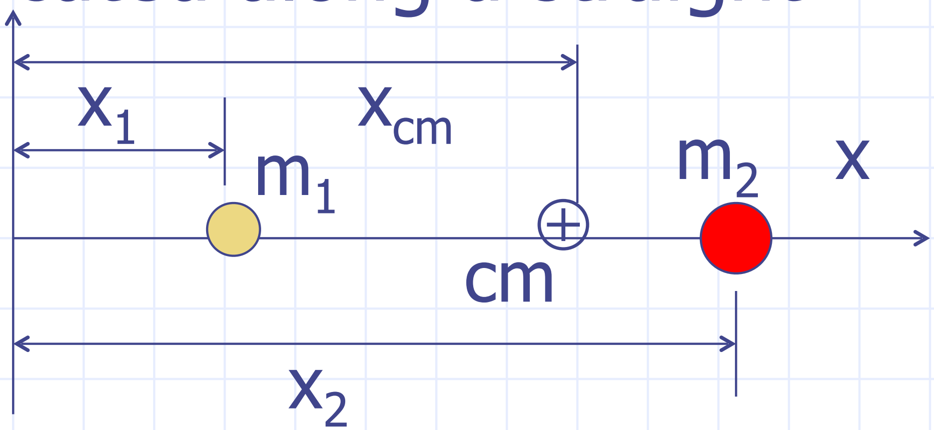
$$m_1 \mathbf{v}_{i1} = m_3 \mathbf{v}_{i3}$$

$$\mathbf{v}_{i1} = \frac{m_3}{m_1} \mathbf{v}_{i3} = \frac{2.5}{4.5} 575 = \boxed{319 \frac{\mathbf{m}}{\mathbf{s}} = \mathbf{v}_{i2}}$$

Center of Mass

- The average location for the total mass of a collection of particles (system)
- Consider two particles located along a straight line

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$



- If $m_1 = m_2 = m$

$$x_{cm} = \frac{mx_1 + mx_2}{m + m} = \frac{m(x_1 + x_2)}{2m} = \frac{x_1 + x_2}{2}$$

- Also can define the center of mass for multiple particles - N number of particles

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \ldots + m_N x_N}{m_1 + m_2 + m_3 + \ldots + m_N}$$

$$x_{cm} = \frac{\sum_{j=1}^{N} m_j x_j}{\sum_{j=1}^{N} m_j} = \frac{\sum_{j=1}^{N} m_j x_j}{M}, \quad M = \sum_{j=1}^{N} m_j$$

Where M is the total mass of the system

- Can also define the center of mass along other coordinate axes - y_{cm} and z_{cm}

- Can also define the velocity of the center of mass

$$\mathrm{v}_{cm} = \frac{m_1\mathrm{v}_1 + m_2\mathrm{v}_2}{m_1 + m_2}, \quad \text{or } \mathrm{v}_{cm} = \frac{\sum_{j=1}^{N} m_j \mathrm{v}_j}{M}$$

$$\text{but } \sum_{j=1}^{N} m_j \mathrm{v}_j = P = \text{total momentum}$$

$$P = M\mathrm{v}_{cm}$$

- If linear momentum is conserved, then

$$P_i = P_f \Rightarrow M\mathrm{v}_{i,cm} = M\mathrm{v}_{f,cm}$$

$$\mathrm{v}_{i,cm} = \mathrm{v}_{f,cm}$$

The velocity of the center of mass is constant. It is the same before and after a collision.

Return to earlier Example

m_1 = 45 kg, m_2 = 12 kg, $\mathbf{v}_{i1}$ = +5.1 m/s, $\mathbf{v}_{i2}$ = 0

We found $\mathbf{v}_f$ = 4.0 m/s

Now, compute the velocity of the center of mass

$$v_{i,cm} = \frac{m_1 v_{i1} + m_2 v_{i2}}{m_1 + m_2} = \frac{45(5.1) + 12(0)}{45 + 12} = \boxed{4.0\,\frac{\mathrm{m}}{\mathrm{s}}}$$

$$v_{f,cm} = \frac{m_1 v_{f1} + m_2 v_{f2}}{m_1 + m_2} = \frac{45(4.0) + 12(4.0)}{45 + 12} = \boxed{4.0\,\frac{\mathrm{m}}{\mathrm{s}}}$$

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{45(0) + 12(\Delta x)}{45 + 12} = \boxed{0.21\Delta x}$$

Taking x_1=0 and $\Delta x = x_2 - x_1 = x_2$